Extra Credit Questions

November 19, 2008

1. Can $d\theta$ be the differential of a function on the (unit) circle? Hint: The case $f(b) - f(a) = \int_a^b f'(x)dx$ of the fundamental theorem of calculus holds when f is a function from the circle to $\mathbb{R}$.

2. Suppose that we are given a function $f(x)$ with domain $\mathbb{R}$. Define $g_a(x) = \int_a^x \frac{-dx}{f'(x)}$. What can be said about the intersections of the graphs of g_a and f and the graph of g_a at the critical points of f? What happens when we vary a?

3. The logistic differential equation is $\frac{dy}{dx} = y(1-y)$. Solve the equation (find y as a function of x) and discuss its history and relation to popluation dynamics.

4. The 4-dimensional ball of radius r can be described as the set of solutions to $x^2+y^2+z^2+w^2 \leq r^2$ and the 3-dimensional sphere can be described as the set of solutions to $x^2+y^2+z^2+w^2 = r^2$. We can think of both as higher dimensional solids/surfaces of revolution. Using this idea, derive formulas for the 4-dimensional volume of the 4-ball and 3-dimensional volume of the 3-sphere in terms of their radius r.

5. Let $\phi(x)$ be 1 for $-1/2 \leq x \leq 1/2$ and 0 for all other x. Given a function f, consider the integral
$$g(x) = \int_{-\infty}^{\infty} f(t)\phi(x-t)dt$$
Interpret g in terms of f. Is g differentiable? Plot g for $f(x) = |x|$ and $f(x) = \sqrt{|x|}$. Hint: This is an example of a convolution integral. It may be useful to plot g using a computer algebra system. How does convolving change the shape of the graph and why?

6. Are there more irrational numbers or rational numbers? Describe what "more" means.

7. Can a continuous function from the line to the square $[0,1] \times [0,1]$ hit every point? Can a differentiable function?