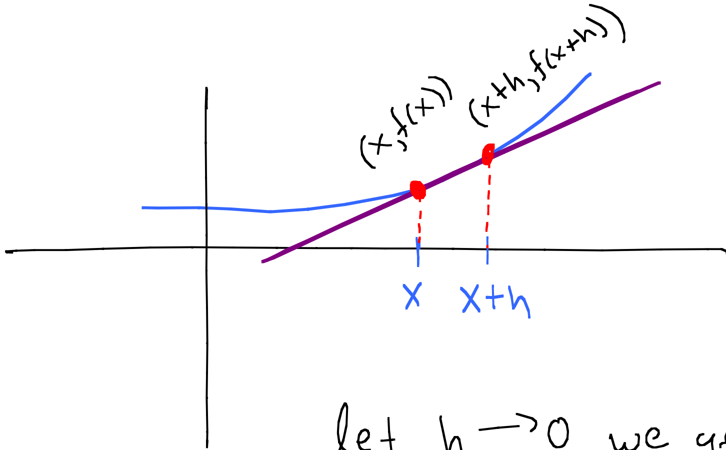
Calc 1 Lecture 7 - Derivatives

Note Title 9/24/2008



let $h \to 0$ we get the tangent line

$$\text{slope} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}$$

Def: The derivative of $f(x)$ at $x=a$, $f'(a)$, is the slope of the tangent line to f at the point $(a, f(a))$

i.e. $f'(a)$ is the instantaneous velocity of f at $x=a$

Another Def: $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$

we also write $\frac{d}{dx}(f(x))$, $\frac{dy}{dx}$, $\frac{df}{dx}$, $Df(x)$, y' for $f'(x)$

Writing $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

we think of f' as a function

Ex: $f(x) = x^2 + \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{\left[(x+h)^2 + \frac{1}{x+h}\right] - \left[x^2 + \frac{1}{x}\right]}{h}$$

change to common denom.

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \frac{1}{x+h} - \cancel{x^2} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + \frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}} - \frac{\cancel{h}}{x(x+h)}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h - \frac{1}{x(x+h)}$$

$$= 2x - \frac{1}{x^2}$$

Ex: $f(x) = x^7 + x^2$

$$f'(x) = \lim_{h \to 0} \frac{\left[(x+h)^7 + (x+h)^2\right] - \left[x^7 + x^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{\begin{pmatrix} \cancel{x^7} + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7 \\ + \cancel{x^2} + 2xh + h^2 \\ - \cancel{x^7} - \cancel{x^2} \end{pmatrix}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(7x^6 + 21x^5h + 35x^4h^2 + 35x^3h^3 + 21x^2h^4 + 7xh^5 + h^6 + 2x + h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 7x^6 + 21x^5h + 35x^4h^2 + 35x^3h^3 + 21x^2h^4 + 7xh^5 + h^6 + 2x + h$$

$$= 7x^6 + 2x$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Ex: $f(x) = \sqrt{x^2+4}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^2+4} - \sqrt{x^2+4}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)^2+4} + \sqrt{x^2+4}}{\sqrt{(x+h)^2+4} + \sqrt{x^2+4}} \cdot \frac{\sqrt{(x+h)^2+4} - \sqrt{x^2+4}}{h}$$

$$= \lim_{h \to 0} [(x+h)^2+4] - [x^2+4] \Big/ h\left(\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right)$$

$$= \lim_{h \to 0} \frac{\cancel{x^2}+2xh+h^2\cancel{+4}\cancel{-x^2}\cancel{-4}}{h\left(\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right)}$$

$$= \lim_{h \to 0} \frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right)}$$

$$= \lim_{h \to 0} \frac{2x+h}{\sqrt{(x+h)^2+4} + \sqrt{x^2+4}}$$

$$= \frac{2x}{2\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+4}}$$

Curve Sketching: $f'(a) > 0 \Leftrightarrow f$ is increasing @ a

$f'(a) < 0 \Leftrightarrow f$ is decreasing @ a

$f'(a) = 0 \Leftrightarrow f$ has horizontal tan. line @ $x = a$

Previous example: $f(x) = \sqrt{x^2+4}$ $f'(x) = \frac{x}{\sqrt{x^2+4}}$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{x^{-1}}{x^{-1}} \frac{x}{\sqrt{x^2+4}}$$

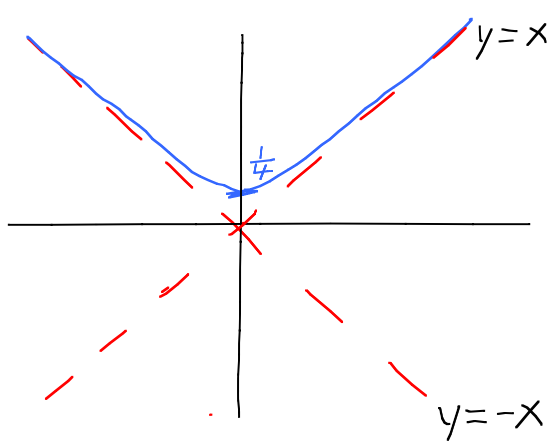
$x > 0 \Rightarrow \sqrt{x^2} = x$

f'

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+4/x^2}}$$

$$= 1$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+4}} = \lim_{x\to-\infty} \frac{-1}{\sqrt{1+4/x^2}} = -1$$



$f'(x) = 0$ $x = ?$

$\frac{x}{\sqrt{x^2+4}} = 0 \Rightarrow x = 0$

$f' > 0$ when $x > 0$

$f' < 0$ when $x < 0$

Ex: $f(x)=|x|$

$$f'(x)=\lim_{h\to 0}\frac{|x+h|-|x|}{h}$$

if $x>0$

$$=\lim_{h\to 0}\frac{x+h-x}{h}=1$$

if $x<0$

$$=\lim_{h\to 0}\frac{-(x+h)-(-x)}{h}=-1$$

if $x=0$

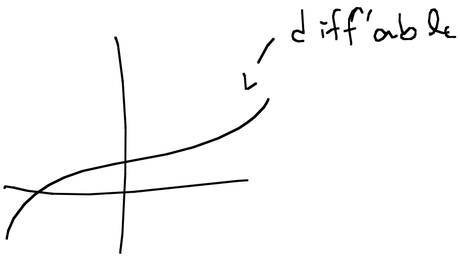
$$=\lim_{h\to 0}\frac{|h|}{h}\quad \boxed{\text{does not exist}}$$

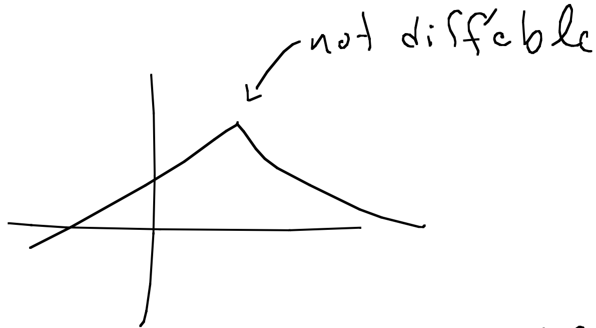
$|x|$ is not differentiable at $x=0$

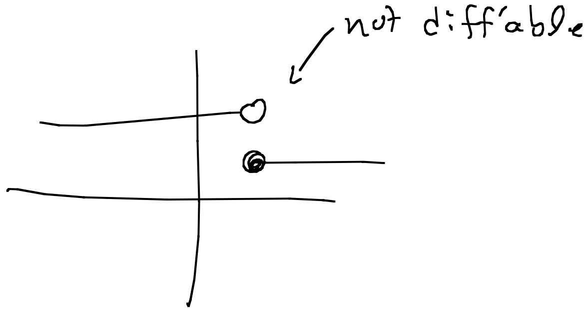
Def: $f(x)$ is differentiable at $x=a$ if $f'(a)$ exists
$f(x)$ is diff'able on (a,b) if $f'(x)$ exists on all $x\in(a,b)$

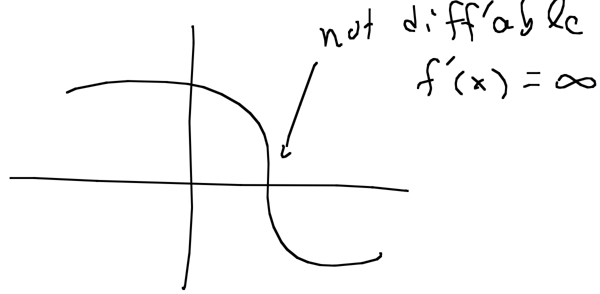
Theorem: If $f(x)$ is diff'able at $x=a$ then $f(x)$ is continuous at $x=a$

(other direction does not work, $|x|$ continuous but not diff'able at $x=0$)









1st Derivative $\quad f'(x) \quad$ Velocity

2nd Derivative $\quad f''(x) = (f')'(x) = \frac{d^2}{dx^2}(f)$

Accelleration

3rd Derivative $\quad f'''(x) = (f'')'(x) = \frac{d^3}{dx^3}(f) \quad$ Jerk

nth Derivative $\quad f^{(n)}(x) = \left(f^{(n-1)}\right)'(x)$

Def: f is a smooth function if all of its derivatives exist and are continuous.