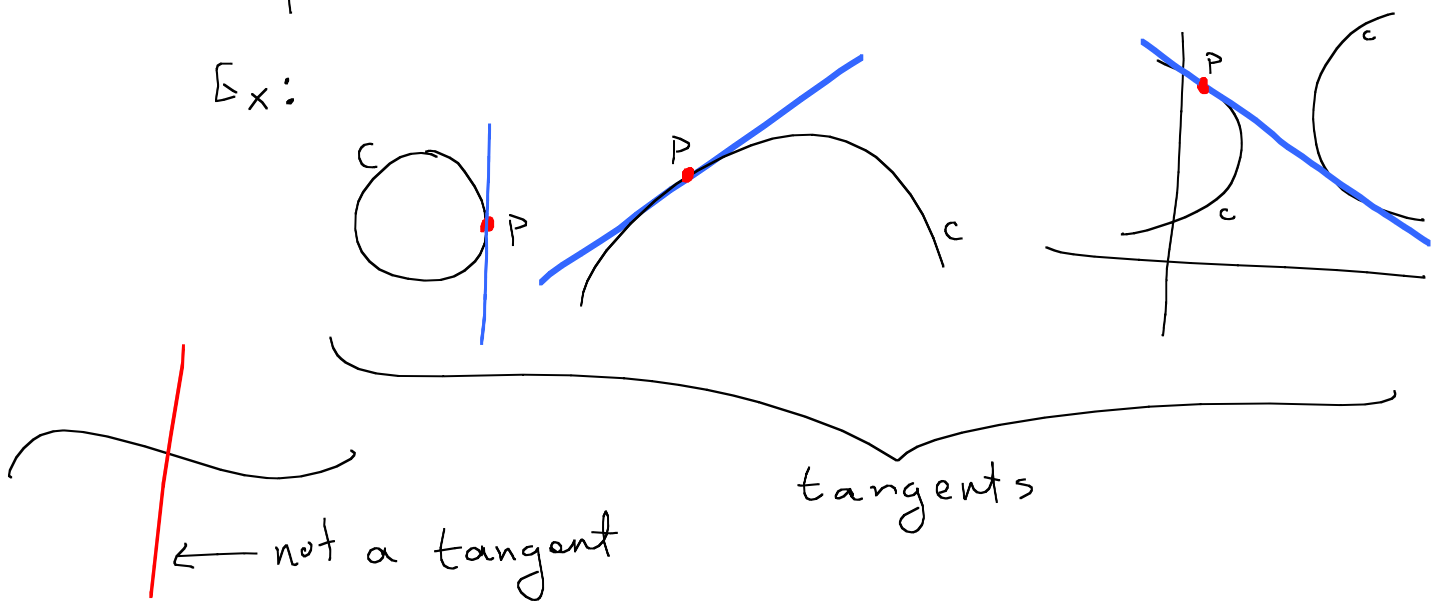
Calc 1 Lecture 4 - Tangents, Limits

Note Title 9/15/2008

Def: The <u>tangent line</u> or simply tangent to a curve C at a point P in C is a line which passes through P and has the same slope as C at P.

Ex:



Notice how similar the curve and tangent line are near P!

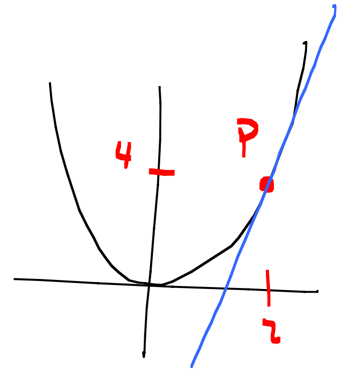
Question: How do we find the equation of the tangent line?

unknown

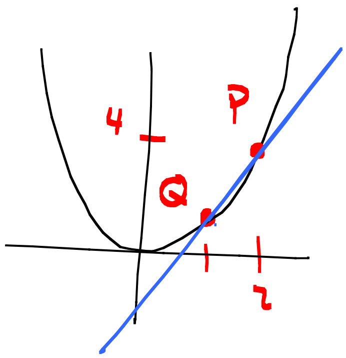
$P = (x_1, y_1)$

$$y = m(x - x_1) + y_1$$

Ex: $y = x^2$ $\quad P = (2,4)$



Approx. the tangent line:



$Q = (1,1) \quad m = \dfrac{4-1}{2-1} = \dfrac{3}{1} = 3$

$Q = \left(\frac{3}{2}, \frac{9}{4}\right) \quad m = \dfrac{4 - \frac{9}{4}}{2 - \frac{3}{2}} = 7/2 = 3.5$

$Q = \left(\frac{5}{3}, \frac{25}{9}\right) \quad m = \dfrac{4 - \frac{25}{9}}{2 - 5/3} = 11/3 \approx 3.66\overline{6}$

$Q = \left(\frac{7}{4}, \frac{49}{16}\right) \quad m = \dfrac{4 - \frac{49}{16}}{2 - 7/4} = 15/4 \approx 3.75$

$Q = (1.99, 3.9601) \quad m = \dfrac{4 - 3.9601}{2. - 1.99} = 3.99$

$Q = (1.999, 3.996001) \quad m = \dfrac{4 - 3.996001}{2 - 1.999} = 3.999$

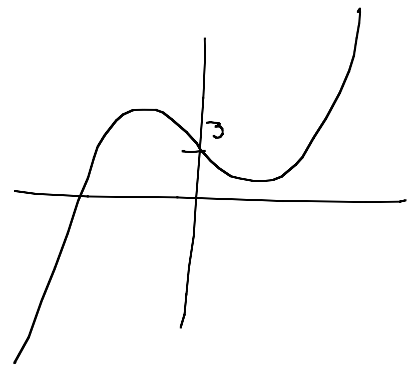
Reasonable Guess for slope: 4

tangent line:

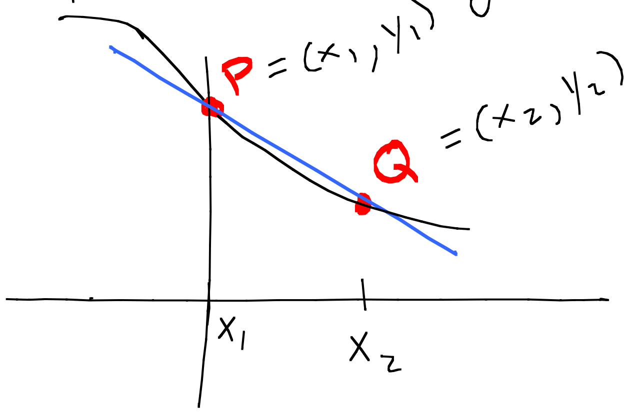
$y = 4(x-2) + 4 = 4x - 4$

Ex: $y = x^3 - 3x + 3$

$P = (0, 3)$



Approximate the tangent line:



Then $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$ is approximation to slope of tan. line

$$\Delta x = x_2 - x_1 \longrightarrow x_2 = x_1 + \Delta x$$

$$\Delta y = \left[(x_1 + \Delta x)^3 - 3(x_1 + \Delta x) + 3\right] - \left[x_1^3 - 3x_1 + 3\right]$$

$$= \cancel{x_1^3} + 3x_1^2 \Delta x + 3x_1 (\Delta x)^2 + (\Delta x)^3 - \cancel{3x_1} - 3\Delta x + \cancel{3} - \cancel{x_1^3} + \cancel{3x_1} - \cancel{3}$$

$$= 3x_1^2 \Delta x + 3x_1 (\Delta x)^2 + (\Delta x)^3 - 3\Delta x$$

$$\frac{\Delta y}{\Delta x} = \frac{3x_1^2 \Delta x + 3x_1 (\Delta x)^2 + (\Delta x)^3 - 3\Delta x}{\Delta x} = 3x_1^2 + \underbrace{3x_1 \Delta x + (\Delta x)^2}_{} - 3$$

When $\Delta x \to 0$, $\dfrac{\Delta y}{\Delta x} \longrightarrow 3x_1^2 - 3$

$x_1 = 0 \Rightarrow \dfrac{\Delta y}{\Delta x} = -3 \Rightarrow$ tangent line at $P = (0, 3)$
$y = -3(x - 0) + 3 = -3x + 3$

Ex (continued):

x	y	$m = 3x^2 - 3$
-2	1	9
-1	5	0
0	3	-3
1	1	0
2	5	9

tangents

@ $(-2,1)$:

$$y = 9(x+2)+1 = 9x + 19$$

@ $(-1,5)$:

$$y = 5$$

@ $(0,3)$:

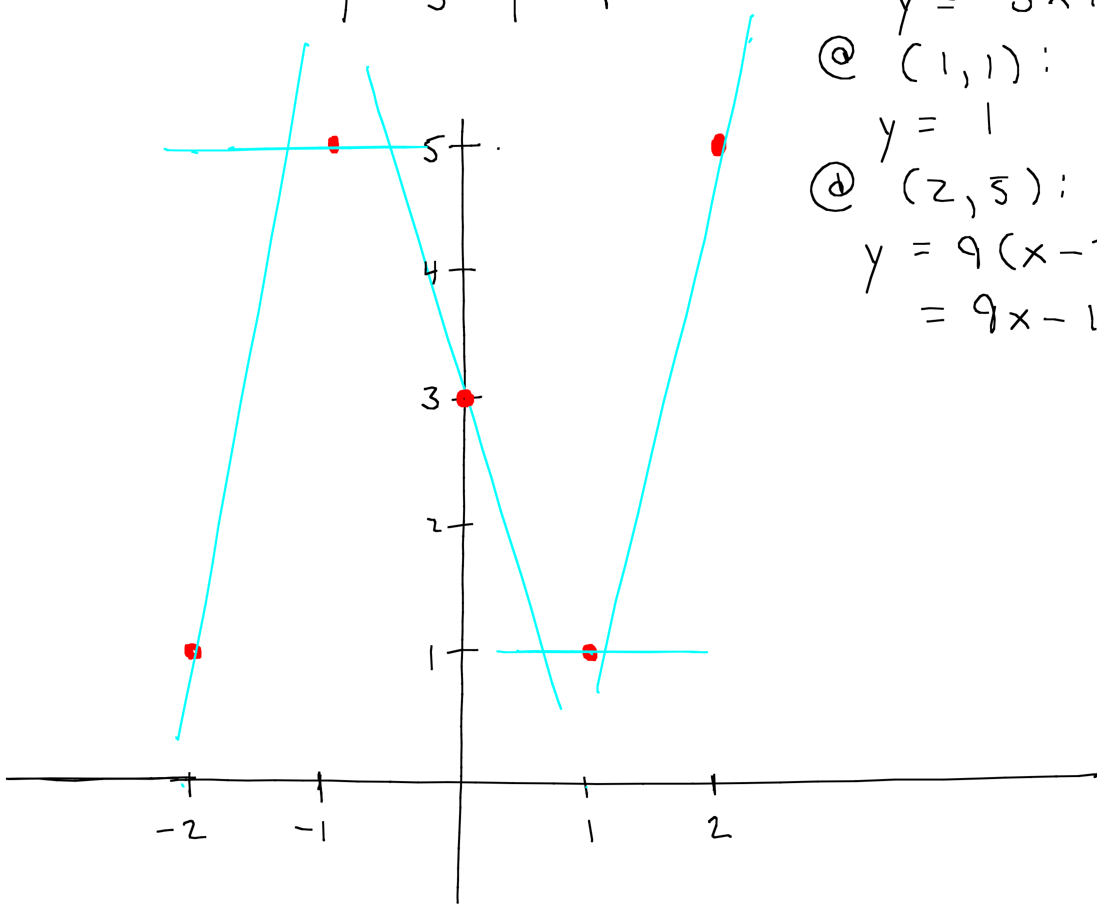
$$y = -3x+3$$

@ $(1,1)$:

$$y = 1$$

@ $(2,5)$:

$$y = 9(x-2)+5 = 9x - 13$$



Instantaneous Velocity:

$$\text{Average velocity} = \frac{\text{Change in Position}}{\text{Change in time}} = \frac{\Delta P}{\Delta t}$$

When Δt very small,

average velocity $\approx$ instantaneous velocity

Ex: Falling in Earth's gravity

$$P(t) = \underbrace{-4.9}_{\text{in } m/s^2} t^2$$

$$\text{Inst. Velocity @ } t = 3 \approx \frac{-46.128 - (-44.1)}{3.1 - 3} = \frac{-2.028}{0.1} = -20.28\ m/s$$

$$\approx \frac{-44.39449 - (-44.1)}{3.01 - 3} = \frac{-0.29449}{0.01} \approx -29.449\ m/s$$

$$\approx \frac{-44.1294049 - (-44.1)}{3.001 - 3} = -29.4049\ m/s$$

Slope of the tangent line to the graph of Position is velocity!

Limits: $\lim_{x \to a} f(x) = L$

"the limit of $f(x)$ as x approaches a is L"

"as $x \to a$, $f(x) \to L$"

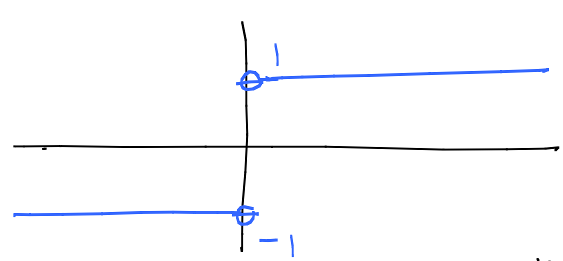
Rough Definition: We say that $\lim_{x \to a} f(x) = L$ when for x-values close to a, $f(x)$ values are close to L

Examples: $\lim_{x \to 7} 2x^2 = 98$

$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ x-1 & \text{if } x > 1 \end{cases}$ then $\lim_{x \to 1} f(x) = 0$

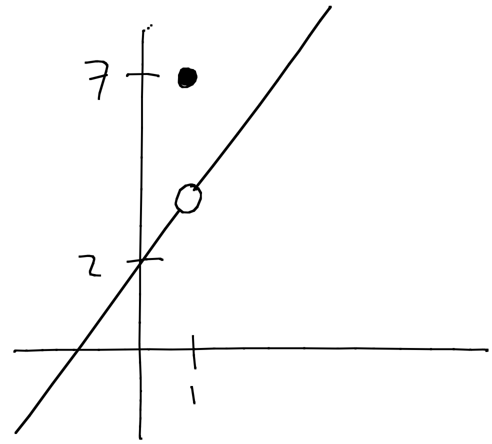
$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

$\lim_{x \to 0} \frac{|x|}{x}$ does not exist



sometimes $\frac{|x|}{x} = 1$, sometimes $\frac{|x|}{x} = -1$ when x near zero

Ex: $f(x) = \begin{cases} x+2 & \text{if } x \neq 1 \\ 7 & \text{if } x = 1 \end{cases}$



$$\lim_{x \to 1} f(x) = 3$$

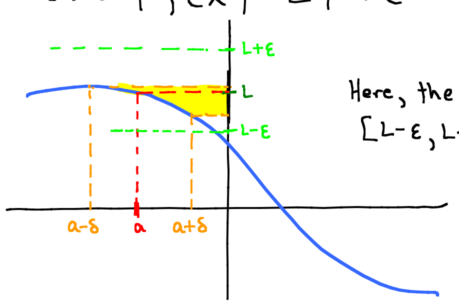
when we approach 1 from the left & right, $f(x)$ goes to 3. It does not matter that $f(1) = 7$.

Optional

Precise Definition of Limit: ε epsilon, δ delta

$\lim_{x \to a} f(x) = L$

when,
for any tolerence for error "ε" that we set,
we can find a distance "δ"
so that:
when $|x-a| < \delta$ (x is less than δ away from a)
we have $|f(x) - L| < \varepsilon$ ($f(x)$ is less than ε away from L)



Here, the yellow region fits inside $[L-\varepsilon, L+\varepsilon]$ so this choice of δ works for this ε.