Calc 1 Lecture 21 - Net Change and the Substitution Rule

Note Title 11/24/2008

Def: Indefinite Integral: $\int f(t)\,dt$ a family of functions which are antiderivatives of f.

We can think of part 2 of the FToC as describing the <u>net change</u> of a function:

$$F(b) - F(a) = \int_a^b F'(t)\,dt$$

Ex: Suppose car brakes produce constant deceleration of K ft/s^2.

What must K be to stop an automobile traveling at 60 mph (88 ft/s) to rest 100 ft from where the brakes were applied?

$a(t) = -K \qquad a(t) = v'(t) \qquad$ (velocity)

$v(t) = \int a(t)\,dt = -Kt + C$

$v(0) = 88 = -K \cdot 0 + C \longrightarrow C = 88$ ft/s

$v(t) = -Kt + 88$

Auto at rest $\Rightarrow v(t) = 0 \quad -Kt + 88 = 0 \quad t = 88/K$

$v(t) = P'(t)$ (position)

$P(t) = \int v(t)\,dt = \frac{K}{2}t^2 + 88t + C$

$P(88/K) = 100$ ft $\quad P(0) = 0 \longrightarrow C = 0$

$\frac{K}{2}\left(\frac{88}{K}\right)^2 + 88\left(\frac{88}{K}\right) + 0 = 100$

over

$$\frac{1}{2}\frac{88^2}{K} + \frac{88^2}{K} = 100 \longrightarrow \frac{3}{2}\frac{88^2}{K} = 100 \longrightarrow \frac{3\cdot 88^2}{200} = K$$

$$K = \frac{2904}{25} = 116.16 \text{ ft/s}^2$$

Substitution Rule:

If $u=g(x)$ is a differentiable function with range an interval I and f is continuous on I then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Ex: $\int \tan x \, dx = \int \frac{\sin x}{\cos x}\,dx$ with $u=\cos x$, $du = -\sin x\,dx$

$$= -\int \frac{-\sin x\,dx}{\cos x} = -\int \frac{du}{u} = -\ln|u| + C$$

$$= \ln|u|^{-1} + C = \ln|u^{-1}| + C$$

$$= \ln|(\cos x)^{-1}| + C$$

$$= \ln|\sec x| + C$$

Ex: $\int \frac{x}{x^2+1}\,dx$ let $u = x^2+1$ so $du = 2x\,dx$

$$= \frac{1}{2}\int \frac{2x\,dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$$

$x^2+1>0$ for all x

$$= \ln\sqrt{x^2+1} + C$$

Ex: $\int x\sin(x^2)\,dx$ $u = x^2$ $du = 2x\,dx$

$$= \frac{1}{2}\int \sin(x^2)\,2x\,dx = \frac{1}{2}\int \sin(u)\,du = \frac{-1}{2}\cos(u) + C$$

$$= -\tfrac{1}{2}\cos(x^2) + C$$

Ex: $\int \cos\theta (\sin\theta)^6 \, d\theta$ $\quad u = \sin\theta \quad du = \cos\theta \, d\theta$

$= \int (\sin\theta)^6 \cos\theta \, d\theta = \int u^6 du = \frac{1}{7} u^7 + C = \frac{1}{7}(\sin\theta)^7 + C$

Ex: $\int x^3 \sqrt{x^2+1} \, dx$ $\quad u = x^2+1 \quad du = 2x$

$= \frac{1}{2} \int x^2 \sqrt{x^2+1} \; 2x \, dx$ $\quad x^2 = u-1$

$= \frac{1}{2} \int (u-1)\sqrt{u} \, du$

$= \frac{1}{2} \int \left(u^{3/2} du - \sqrt{u} \, du \right) = \frac{1}{2} \int u^{3/2} du - \frac{1}{2} \int \sqrt{u} \, du$

$= \frac{1}{2} \cdot \frac{2}{5} u^{5/2} - \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$

$= \frac{1}{5} u^{5/2} - \frac{1}{3} u^{3/2} + C$

$= \frac{1}{5} (x^2+1)^{5/2} - \frac{1}{3} (x^2+1)^{3/2} + C$

Substitution Rule for Definite Integrals:

If g' continuous on $[a,b]$ and f continuous on the range of $u=g(x)$ then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Ex: $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$ $\qquad u=\ln x \quad du=\frac{dx}{x}$

$u(e^4)=4 \quad u(e)=1$

$$=\int_e^{e^4} \frac{1}{\sqrt{\ln x}}\,\frac{dx}{x} = \int_1^4 \frac{1}{\sqrt{u}}\,du = \int_1^4 u^{-1/2}\,du$$

$$= 2u^{1/2}\Big|_1^4 = 2\sqrt{4}-2\sqrt{1}=2$$

Ex: $\displaystyle\int_0^1 x e^{-x^2}\,dx$ $\qquad u=-x^2 \quad du=-2x\,dx$

$u(1)=-1 \quad u(0)=0$

$$=-\frac{1}{2}\int_0^1 e^{-x^2}(-2x)\,dx = -\frac{1}{2}\int_0^{-1} e^u\,du$$

$$=\frac{1}{2}\int_{-1}^0 e^u\,du = \frac{1}{2}e^u\Big|_{-1}^0 = \frac{1}{2}e^0-\frac{1}{2}e^{-1}$$

$$=\frac{1}{2}-\frac{1}{2e}$$

Ex: $\displaystyle\int_0^\pi \sec^2(t/4)\,dt$ $\qquad u=t/4 \quad du=1/4\,dt$

$u(\pi)=\pi/4 \quad u(0)=0$

$$=4\int_0^\pi \sec^2(t/4)\,\frac{1}{4}dt$$

$$=4\int_0^{\pi/4}\sec^2(u)\,du = 4\tan(u)\Big|_0^{\pi/4} = 4(\tan\tfrac{\pi}{4})-4(\tan 0)$$

$$=4$$