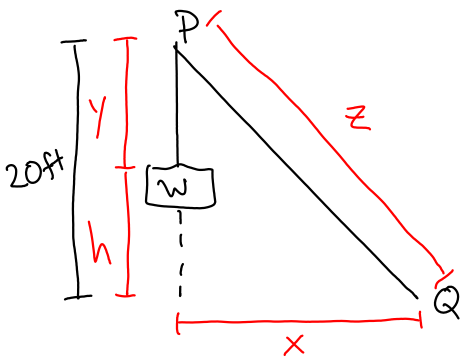
Calc I Lecture 12 - Related Rates & Hyperbolic Functions

Related Rates



P - pulley W - weight Q - you

45 foot long cable

$y + z = 45$ cable length
$h + y = 20$ Pulley height
$20^2 + x^2 = z^2$ Pythagorean Theorem

x, y, z, h all vary with time so think of them as functions of t

Q: What is $\frac{dh}{dt}$ when $x = 15$ ft, $\frac{dx}{dt} = 6 \frac{ft}{s}$?

Reduce # of variables: $z = 45 - y$, $y = 20 - h \longrightarrow z = 25 + h$

$$400 + x^2 = (25+h)^2$$

Diff with respect to t:

$$\frac{d}{dt}(400 + x^2) = \frac{d}{dt}\left[(25+h)^2\right]$$

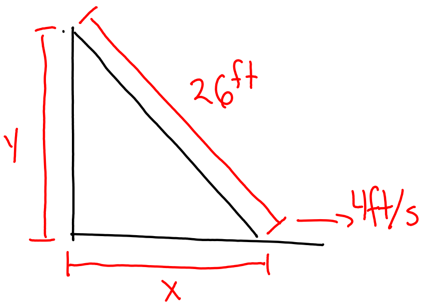
$$2x \frac{dx}{dt} = 2(25+h)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{x}{25+h}\, dx/dt$$

when $x = 15$, $400 + 15^2 = (25+h)^2 \longrightarrow h = 0$

So when $x = 15$ & $\frac{dx}{dt} = 6$, $\frac{dh}{dt} = \frac{15}{25+0} \cdot 6 = \frac{18}{5} = 3.6$ ft/s

Another Example:



A 26ft long ladder leans against a wall. If the base of the ladder is 10 ft away from the wall and we the base away from the wall at a rate of 4 ft/s, how fast is the top of the ladder dropping?

x, y functions of t $\qquad x^2+y^2=26^2$

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(26)$$
$$2x\,dx/dt+2y\,dy/dt=0$$

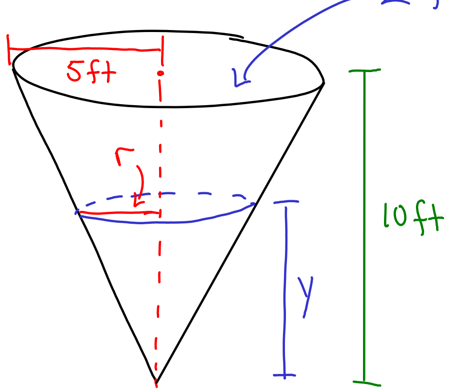
if $x=10$, $10^2+y^2=26^2$
$$y^2=676-100$$
$$y=24$$

$$\frac{dy}{dt}=\frac{-x}{y}\frac{dx}{dt}$$

when $x=10$, $y=24$, $\frac{dx}{dt}=4$ then $\frac{dy}{dt}=\frac{-10}{24}\cdot 4=-5/3$ ft/s

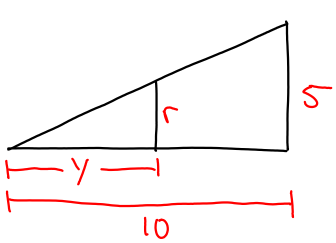
Another Ex:

2 ft³/s water in



How fast is the water rising when the existing water is 6ft deep?

$V=\frac{1}{3}\pi r^2 y$ Volume of water



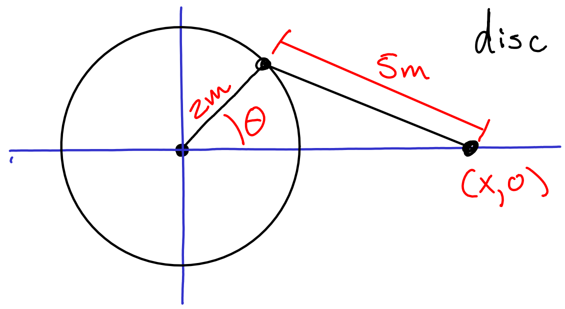
so $\frac{r}{y}=\frac{5}{10} \longrightarrow r=\frac{1}{2}y$

$$V=\frac{\pi}{3}\left(\frac{1}{2}y\right)^2 y$$
$$V=\frac{\pi}{12}y^3$$

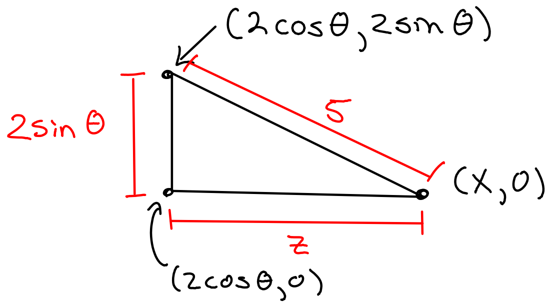
Diff w.r.t. t

$$\frac{dV}{dt}=\frac{\pi}{12}3y^2\frac{dy}{dt} \longrightarrow \frac{dy}{dt}=\frac{4}{\pi y^2}\frac{dV}{dt}=\frac{4}{\pi 6^2}2$$
$$=\frac{2}{9\pi}\text{ ft/s}\approx 0.071\text{ ft/s}$$

Another Ex: An arm of length 5 m attached to a rotating disc of radius 2 m and the x-axis.



What is $\frac{dx}{dt}$ when $\theta = \frac{\pi}{4}$ and $\frac{d\theta}{dt} = 2\pi$ rad/sec ?



$$(2\sin\theta)^2 + z^2 = 5^2$$

$$x = 2\cos\theta + z$$

$$\frac{dx}{dt} = -2\sin\theta \frac{d\theta}{dt} + \frac{dz}{dt}$$

$$2(2\sin\theta)2\cos\theta\frac{d\theta}{dt} + 2z\frac{dz}{dt} = 0$$

$$\frac{dz}{dt} = \frac{4\sin\theta\cos\theta}{z}\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \left(-2\sin\theta + \frac{4\sin\theta\cos\theta}{z}\right)\frac{d\theta}{dt}$$

when $\theta = \frac{\pi}{4}$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and

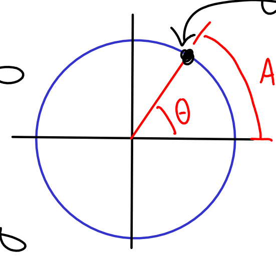
$$\left(2\frac{\sqrt{2}}{2}\right)^2 + z^2 = 25$$

$$z = \sqrt{23}$$

$$\frac{dx}{dt} = \left(-2\frac{\sqrt{2}}{2} + \frac{4\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}}{\sqrt{23}}\right)2\pi = \left(\frac{2}{\sqrt{23}} - \sqrt{2}\right)2\pi \quad m/s$$

Hyperbolic Trig Functions

Regular Trig

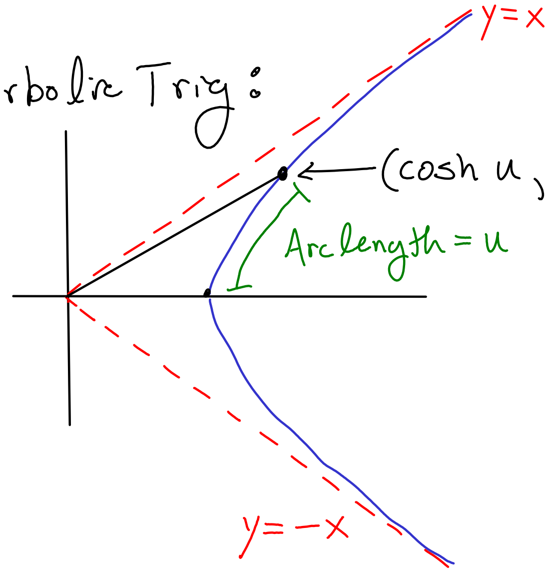


if unit circle, $r=1$ so $r\theta = \theta$

$x^2 + y^2 = 1$

Hyperbolic Trig:

unit hyperbola $x^2 - y^2 = 1$



$$\sinh(u) = \frac{e^u - e^{-u}}{2}$$

$$\cosh(u) = \frac{e^u + e^{-u}}{2}$$

$$\tanh(u) = \frac{\sinh(u)}{\cosh(u)} \qquad \operatorname{csch}(u) = \frac{1}{\sinh(u)} \qquad \operatorname{sech}(u) = \frac{1}{\cosh(u)}$$

$$\coth(u) = \frac{1}{\tanh(u)} = \frac{\cosh(u)}{\sinh(u)}$$

Identities: $\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x)$

$$(\cosh(x))^2 - (\sinh(x))^2 = 1 \qquad 1 - (\tanh x)^2 = (\operatorname{sech}(x))^2$$

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

$\frac{d}{dx}(\sinh(x)) = \cosh(x) \qquad \frac{d}{dx}(\cosh(x)) = \sinh(x)$ (Notice the sign)

$\frac{d}{dx}(\tanh(x)) = [\operatorname{sech}(x)]^2 \qquad \frac{d}{dx}(\coth(x)) = -(\operatorname{csch}(x))^2$

$\frac{d}{dx}(\operatorname{csch}(x)) = -\operatorname{csch}(x)\coth(x) \qquad \frac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech}(x)\tanh(x)$

$$\left.\begin{aligned} \sinh^{-1}(x) &= \ln\left(x+\sqrt{x^2+1}\right) & x \in \mathbb{R} \\ \cosh^{-1}(x) &= \ln\left(x+\sqrt{x^2-1}\right) & x \geq 1 \\ \tanh^{-1}(x) &= \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) & -1 < x < 1 \end{aligned}\right\} \text{Inverses}$$

$$\frac{d}{dx}\left(\sinh^{-1}(x)\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\cosh^{-1}(x)\right) = \frac{1}{\sqrt{x^2-1}}$$

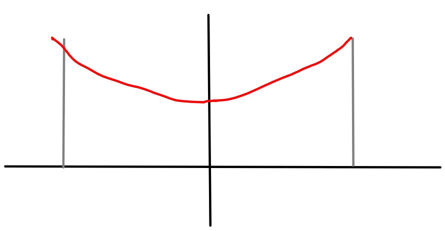
$$\frac{d}{dx}\left(\tanh^{-1}(x)\right) = \frac{1}{1-x^2} \quad \text{when } |x| < 1$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1}(x)\right) = \frac{-1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1}(x)\right) = \frac{-1}{x\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left(\coth^{-1}(x)\right) = \frac{1}{1-x^2} \quad \text{when } |x| > 1$$

Catenary:

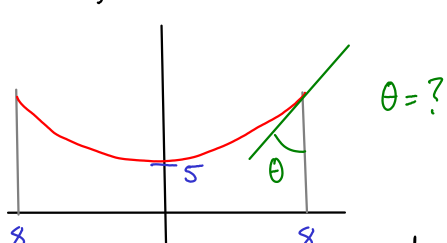


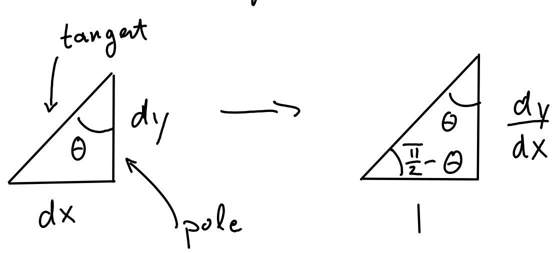
Hanging wire hangs in the form

$$y = c + a\cosh\left(\frac{x}{a}\right)$$

Ex: At what angle does the wire hit the pole?

$$y = 10\cosh\left(\frac{x}{10}\right) - 5$$





$$\frac{\pi}{2} - \theta = \arctan\left(\frac{dy}{dx}\right)$$

$$\theta = \frac{\pi}{2} - \arctan\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(10\cosh\left(\frac{x}{10}\right) - 5\right) = \sinh\left(\frac{x}{10}\right) \to \theta = \frac{\pi}{2} - \arctan\left(\sinh\left(\frac{8}{10}\right)\right)$$