Calc 1 Lecture 11 - More Logarithmic Derivatives and Implicit Differentiation

Note Title 10/13/2008

Logarithmic Derivatives:

$$\frac{d}{dx}\left(\ln|f(x)|\right) = \frac{f'(x)}{f(x)}$$ by Chain Rule, when $f(x) \neq 0$

so $$f'(x) = f(x) \cdot \frac{d}{dx}\left(\ln|f(x)|\right)$$

Ex: $f(x) = x^{1/x}$

$$\frac{d}{dx}\left(\ln\left|x^{1/x}\right|\right) = \frac{d}{dx}\left(\ln|x|^{1/x}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\ln|x|\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\right)\ln|x| + \frac{1}{x}\frac{d}{dx}\left(\ln|x|\right)$$

$$= \frac{-\ln|x|}{x^2} + \frac{1}{x^2}$$

$$= \frac{1-\ln|x|}{x^2}$$

$$f'(x) = x^{\frac{1}{x}}\underbrace{\left(\frac{1-\ln|x|}{x^2}\right)}$$

Chain Rule: $f = g \circ h \longrightarrow f' = (g' \circ h) \cdot h'$

Ex: $f(x) = \ln\left(\sqrt{1+e^x}\right)$

$$f'(x) = \frac{1}{\sqrt{1+e^x}} \cdot \frac{d}{dx}\left(\sqrt{1+e^x}\right)$$

$$= \frac{1}{\sqrt{1+e^x}} \cdot \frac{1}{2}(1+e^x)^{-1/2} \frac{d}{dx}(1+e^x)$$

$$= \frac{1}{\sqrt{1+e^x}} \cdot \frac{1}{2\sqrt{1+e^x}} \cdot e^x$$

$$= \frac{e^x}{2+2e^x}$$

Implicit Differentiation:

How do we differentiate:

defines a circle of radius $\sqrt{4} = 2$

$$y = x^2 + 2x \qquad\qquad x^2 + y^2 = 4$$

y is a function of x
i.e. $y = f(x) = x^2 + 2x$
find $f'(x)$ regular way

solve for y as a function of x
$y = \pm\sqrt{4 - x^2}$
& differentiate

OR

Implicit differentiation

With Implicit Differentiation, we assume that y is a function of x, without knowing the formula.

$$\frac{d}{dx}(y) = \frac{dy}{dx} \qquad\qquad \underbrace{\frac{d}{dx}\left(y^3\right) = 3y^2\frac{dy}{dx}}_{\text{chain Rule}}$$

Ex:

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(4)$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=0$$

$$2x\underbrace{\frac{dx}{dx}}_{1}+2y\frac{dy}{dx}=0$$

Solve for $\frac{dy}{dx}$: $\quad \frac{dy}{dx}=\frac{-x}{y}$

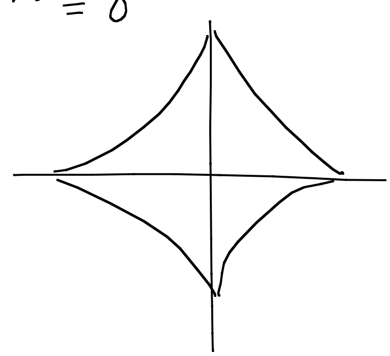
Ex: $x^2-y^2=1$ (unit hyperbola)

$$\frac{d}{dx}\left(x^2-y^2\right)=\frac{d}{dx}(1)$$

$$2x-2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{x}{y}$$

Ex: Find All points (x,y) on $x^{2/3}+y^{2/3}=8$ where the tangent line has slope $=-1$



$$\frac{d}{dx}\left(x^{2/3}+y^{2/3}\right)=\frac{d}{dx}(8)$$

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$$

solve: $-1=\sqrt[3]{\frac{y}{x}}$

$-1=y/x \implies y=-x$

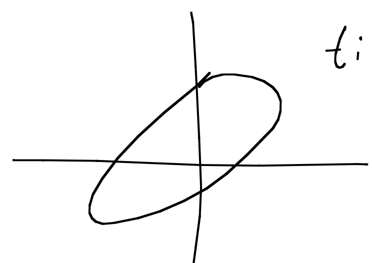
from previous:

$y = -x$ on $x^{2/3} + y^{2/3} = 8$

$$2x^{2/3} = 8$$
$$x^{2/3} = 4$$
$$x^{1/3} = \pm 2$$
$$x = \pm 8 \quad \text{so} \quad y = \mp 8$$

points that have a tangent line with slope $= -1$: $(8,-8)$, $(-8,8)$

Ex: $x^2 - xy + y^2 = 3$



tilted ellipse

largest & smallest values of y?

occurs at $\frac{dy}{dx} = 0$

$$\frac{d}{dx}\left(x^2 - xy + y^2\right) = \frac{d}{dx}(3)$$

$$2x - \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 3$$

$$2x - y = 3$$
$$y = 2x - 3$$

$$x^2 - xy + y^2 = 3$$
$$x^2 - x(2x-3) + (2x-3)^2 = 3$$
$$x^2 - 2x^2 + 3x + 4x^2 - 12x + 9 = 3$$
$$3x^2 - 9x + 6 = 0$$
$$x^2 - 3x + 2 = 0$$
$$(x-2)(x-1) = 0$$

from previous:

$x = 1$ or $x = 2$ and $y = 2x - 3$

so $\underbrace{y = 2\cdot 1 - 3 = -1}_{\text{min}}$ or $\underbrace{y = 2\cdot 2 - 3 = 1}_{\text{max}}$

$y = \sin^{-1} x \longrightarrow x = \sin y$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = \cos(y) \cdot \frac{dy}{dx}$$

$$\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$$

so $\frac{dy}{dx} = 1 / \sqrt{1 - x^2}$

$$\frac{d}{dx}(\sin^{-1} x) = 1 / \sqrt{1 - x^2}$$

$$y = \tan^{-1} x \longrightarrow x = \tan y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$

$$1 = \sec^2(y) \cdot \frac{dy}{dx}$$

$$\sec^2(y) = 1 + \tan^2(y) = 1 + x^2$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{dy}{dx} = 1/1+x^2$$

Dictionary:

$$\frac{d}{dx}(\cos^{-1}(x)) = -1/\sqrt{1-x^2}$$

$$\frac{d}{dx}(\csc^{-1}(x)) = -1/x\sqrt{x^2-1}$$

$$\frac{d}{dx}(\sec^{-1}(x)) = 1/x\sqrt{x^2-1}$$

$$\frac{d}{dx}(\cot^{-1}(x)) = -1/1+x^2$$