Calc 1 Lecture 10 - The Chain Rule & Logarithms

Note Title 10/8/2008

Question: How do we differentiate functions such as

$$f(x) = \sin(\sqrt{x^2+1})$$

No previous rule works. Notice that f is formed by composing a chain of functions:

$$g(x) = x^2+1$$
$$h(x) = \sqrt{x}$$
$$i(x) = \sin x$$

so $f(x) = i(h(g(x)))$

To find the derivative, we need "the Chain Rule".

Theorem: (Chain Rule) If $f(x) = g(K(x))$, K is differentiable at x and g differentiable at $K(x)$ then

$$f'(x) = g'(K(x)) \cdot K'(x)$$

Proof:

$$f'(x) = \lim_{h \to 0} \frac{g(K(x+h)) - g(K(x))}{h}$$

$$= \lim_{h \to 0} \frac{g(K(x+h)) - g(K(x))}{K(x+h) - K(x)} \cdot \frac{K(x+h) - K(x)}{h}$$

$$= \lim_{h \to 0} \frac{g(K(x+h)) - g(K(x))}{K(x+h) - K(x)} \cdot \lim_{h \to 0} \frac{K(x+h) - K(x)}{h}$$

let $u = K(x)$, $v = K(x+h)$

$$= \left(\lim_{v \to u} \frac{g(v) - g(u)}{v - u} \right) \cdot K'(x)$$

$$= g'(u) \cdot K'(x)$$

$$= g'(K(x)) K'(x) \quad \blacksquare$$

We can also think that if $y = f(x) = g(K(x))$ and $u = K(x)$

then $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Examples:

1) $y = \sqrt{x^3+2}$ $\qquad u = x^3+2 \rightsquigarrow y = \sqrt{u}$

$$\frac{dy}{dx} = \frac{1}{2}(u)^{-1/2}\frac{du}{dx}$$

$$= \frac{1}{2}u^{-1/2}(3x^2)$$

$$= \frac{1}{2}(x^3+2)^{-1/2}(3x^2) = \frac{3x^2}{2\sqrt{x^3+2}}$$

2) $y = \sin(x^2)$ $\qquad u = x^2 \rightsquigarrow y = \sin(u)$

$$\frac{dy}{dx} = \cos(u)\frac{du}{dx} = \cos(u)(2x)$$

$$= \cos(x^2)\cdot 2x$$

3) $y = e^{-x^2}$ $\qquad u = -x^2 \rightsquigarrow y = e^u$

$$\frac{dy}{dx} = e^u\frac{du}{dx} = e^u(-2x)$$

$$= -2x\, e^{-x^2}$$

Power Rule + Chain Rule: $\frac{d}{dx}\left((f(x))^n\right) = n(f(x))^{n-1}\cdot f'(x)$

Ex: $\frac{d}{d\theta}\left((\sin\theta)^2\right) = 2\sin\theta\cdot\cos\theta$

Ex (triple Chain Rule)

$y = f(x) = \sin\left(\sqrt{x^2+1}\right)$

$u = \sqrt{x^2+1} \rightsquigarrow f(x) = \sin(u)$

But need the Chain Rule for u as well

$v = x^2+1 \rightsquigarrow u = \sqrt{v}$

So $f'(x) = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

$= \cos(u) \cdot \frac{1}{2} v^{-1/2} \cdot (2x)$

$= \cos\left(\sqrt{x^2+1}\right) \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$

$\frac{d}{dx}(a^x) = \ln(a) \cdot a^x$

Proof: $a^x = \left(e^{\ln(a)}\right)^x = e^{\ln(a)x}$

$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{\ln(a)x}\right) = e^{\ln(a)x} \cdot \frac{d}{dx}(\ln(a)x)$

$= e^{\ln(a)x} \cdot \ln(a) = \ln(a) \cdot a^x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Proof: $e^{\ln(x)} = x$ (Inverse functions)

$$\frac{d}{dx}\left(e^{\ln(x)}\right) = \frac{d}{dx}(x)$$

$$e^{\ln(x)} \frac{d}{dx}(\ln(x)) = 1$$

$$x \frac{d}{dx}(\ln(x)) = 1$$

$$\frac{d}{dx}(\ln(x)) = 1/x$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln(a)}$$

Proof: $\log_a x = \frac{\ln(x)}{\ln(a)}$ so $\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right)$

$$= \frac{1}{\ln(a)} \frac{d}{dx}(\ln(x))$$

$$= \frac{1}{\ln(a)} \cdot \frac{1}{x} = \frac{1}{x \cdot \ln(a)}$$

Note that $\text{Domain}(\ln(x)) = \{x \in \mathbb{R} \mid x > 0\}$
but $\text{Domain}(\ln|x|) = \mathbb{R} \setminus \{0\}$ otherwise same fn.

$$\frac{d}{dx}\left(\ln|x|\right) = 1/x$$

Proof: if $x > 0$, $|x| = x$ so $\ln|x| = \ln x$
and the previous rule holds

if $x < 0$, $|x| = -x$ so $\ln|x| = \ln(-x)$

$$\frac{d}{dx}\left(\ln(-x)\right) = \frac{1}{-x} \frac{d}{dx}(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x} \quad \blacksquare$$

Logarithmic Differentiation (If time permits)

$$\frac{d}{dx}\left[\ln|f(x)|\right] = \frac{f'(x)}{f(x)} \qquad \text{when } f(x) \neq 0$$

so $f'(x) = f(x) \cdot \frac{d}{dx}\left(\ln|f(x)|\right)$

Ex $\quad y = x^x$

$$\frac{d}{dx}\left(\ln|x^x|\right) = \frac{d}{dx}\left(x \ln|x|\right) = \frac{d}{dx}(x)\ln|x| + x\frac{d}{dx}(\ln|x|) = \ln|x| + x \cdot \frac{1}{x} = 1 + \ln|x|$$

so $\frac{dy}{dx} = x^x\left(1 + \ln|x|\right)$

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

Proof when n is not an integer:

$$y = x^n$$

$$\frac{d}{dx}\left(\ln|x^n|\right) = \frac{d}{dx}\left(n\ln|x|\right) = n\frac{d}{dx}\left(\ln|x|\right) = \frac{n}{x}$$

$$\frac{dy}{dx} = x^n \cdot \frac{d}{dx}\left(\ln|x^n|\right) = x^n \cdot \frac{n}{x} = n\,x^{n-1}$$

Ex! $y = (\sin x)^{x^2}$

$$\frac{d}{dx}\left\{\ln\left|(\sin x)^{x^2}\right|\right\} = \frac{d}{dx}\left\{x^2\ln|\sin(x)|\right\}$$

$$= 2x\ln|\sin x| + x^2\frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = (\sin x)^{x^2}\left[2x\ln|\sin x| + \frac{x^2}{\tan x}\right]$$

Ex: $y = (x^2+1)^x$

$$\frac{d}{dx}\left\{ \ln\left| (x^2+1)^x \right| \right\} = \frac{d}{dx}\left\{ x \ln\left| x^2+1 \right| \right\}$$

$$= 1 \cdot \ln\left| x^2+1 \right| + x \cdot \frac{2x}{x^2+1}$$

$$\frac{dy}{dx} = (x^2+1)^x \left[\ln\left| x^2+1 \right| + \frac{2x^2}{x^2+1} \right]$$

$x^2+1 > 0$ for all x so

$$\frac{dy}{dx} = (x^2+1)^x \left[\ln(x^2+1) + \frac{2x^2}{x^2+1} \right]$$