Topology, fall 2022.

Quiz 1 Solutions

I (11 points). Mark the boxes that are followed by correct statements.

1) ■ Collection of sets \( \{\emptyset, \{b\}, \{a, b, c\}\} \) is a topology on the set \( \{a, b, c\}\).
   **True.** This set is closed under finite intersections and arbitrary unions, contains the empty set and the entire space.

2) ■ If \( B_1, B_2 \) are both bases for a topology \( T \) on \( X \) then their union \( B_1 \cup B_2 \) is also a basis for the topology \( T \).
   **True.** This is a good exercise. Check basis axioms for \( B_1 \cup B_2 \).

3) ■ If \( X, Y \) have indiscrete topologies, the product topology on \( X \times Y \) is indiscrete.
   **True.** Use the definition of the product topology to check this.

4) □ Indiscrete topology is finer than any other topology on a set \( X \).
   **False.** It’s the opposite, in fact. The indiscrete topology is coarser than any topology on \( X \).

5) ■ The ordered square \( I^2_o \) is Hausdorff.
   **True.** We proved that any order topology is Hausdorff.

6) ■ A finite topological space is Hausdorff if and only if it is discrete (carries discrete topology).
   **True.** We discussed this briefly in class. A topology is \( T_1 \) if points are closed. In a finite topological space points are closed iff \( X \) is discrete (since then any subset \( Y \subset X \) is closed).

7) □ Set \( \mathbb{N} \) of natural numbers with the finite complement topology is Hausdorff.
   **False.** The finite complement topology on an infinite set is not...
Hausdorff.

8) ■ If $X$ is a metric space, any subset $Y \subset X$ inherits a metric from $X$.
   **True.**

9) □ The interval $[0, \pi]$ with the distance function $d(a, b) = \sin |a - b|$ is a metric space.
   **False.** What is the distance $d(0, \pi)$ in this topology?
   **Suppose you restrict to the open interval $(0, \pi)$. Does that distance function define a metric?**

10) ■ Topological space $X = \{a, b, c\}$ with the topology
    \{\emptyset, \{c\}, \{b, c\}, X\} is connected.
    **True.** There exists no separation of $X$.

11) □ If both $X \cup Y$ and $Y$ are connected then $X$ is connected.
    **False.** For a counterexample, take a connected $Y$ and $X \subset Y$ not connected. Say $Y = \mathbb{R}$ and $X = \{0, 1\}$.

II (5 points) Mark the square in the first column, respectively second column, if the corresponding subset of $\mathbb{R}^2$ with the standard topology is open, respectively closed.

- □ ■ $\{(x, y) | x \geq 0 \text{ or } y \geq 0\}$
- □ □ $\{(x, y) | x < 0 \text{ and } y \geq 0\}$
- □ ■ $\{(x, y) | x = 1 \text{ and } y \leq 2\}$
- □ ■ $\{(x, y) | xy = 1\}$
- □ ■ $\{(x, y) | x^2 + y^2 \geq 1\}$

III (3 points) Mark the boxes that are followed by correct statements.

a) ■ Sequence $x_n = (\frac{1}{n}, \frac{1}{2n}, \frac{1}{3n}, \ldots)$ converges to $0 = (0, 0, 0, \ldots)$
in the uniform topology on $\mathbb{R}^N$. 
**True.** Take a basis neighbourhood $B(\vec{0}, \epsilon)$, $\epsilon > 0$ and check that all $x_n$ starting with some $n$ are in that neighbourhood.

b) $\square$ The identity map $\mathbb{R} \rightarrow \mathbb{R}_{\ell}$ from $\mathbb{R}$ with the standard topology to $\mathbb{R}$ with the lower limit topology is continuous. 
**False.** $\mathbb{R}_{\ell}$ is strictly finer than the standard topology and has more open sets, so that not every inverse image of an open set is open.

c) $\square$ The identity map $\mathbb{R}^N \rightarrow \mathbb{R}^N$ from $\mathbb{R}^N$ with the box topology to $\mathbb{R}^N$ with the product topology is continuous. Here each $\mathbb{R}$ in this infinite product carries the standard topology. 
**True.** The box topology is finer than the product topology.