SACKLER LECTURES

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Lecture 1. Tensor categories.

1.1. Monoidal categories.

(1.1.1) A *monoidal category* is the following collection of data:
(i) A category C.
(ii) A functor $F : C \times C \to C$. Notation: $(X, Y) \mapsto X \otimes Y$
(iii) An associativity constraint, i.e. a system of isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ for any $X, Y, Z \in C$. The system of isomorphisms $a_{X,Y,Z}$ should be functorial in X, Y, Z and satisfy the pentagon identity (PI):

$$(PI) \qquad \begin{array}{ccccc} (X \otimes (Y \otimes Z)) \otimes T & & \longrightarrow & & X \otimes ((Y \otimes Z) \otimes T) \\ \downarrow & & & & \downarrow \\ ((X \otimes Y) \otimes Z) \otimes T & \to & (X \otimes Y) \otimes (Z \otimes T) & \leftarrow & X \otimes (Y \otimes (Z \otimes T)) \end{array}$$

This notion is a generalization of a notion of monoid.

(1.1.2) We usually assume the existence of a unit object. A *unit object* is an object $\mathbb{1}$ together with functorial isomorphisms $\alpha_X : X \xrightarrow{\sim} X \otimes \mathbb{1}$ and $\beta_X : X \to \mathbb{1} \otimes X$, compatible with the associativity constraint.

(1.1.3) In the classical theory most important is the notion of a group, i.e. a monoid in which all elements are invertible. Given an object X in a monoidal category one can define a notion of an inverse object X^{-1}, but this notion is rarely useful. There exists, however, a weaker notion of the same type which *is* useful.

Definition. We say that an object $X \in C$ is *left rigid* if there exist $Y \in C$ and morphisms

$$e : Y \otimes X \to \mathbb{1}$$
$$i : \mathbb{1} \to X \otimes Y$$

such that the compositions

$$X \to \mathbb{1} \otimes X \to X \otimes Y \otimes X \to X \otimes \mathbb{1} \to X$$

$$Y \to Y \otimes \mathbb{1} \to Y \otimes X \otimes Y \to \mathbb{1} \otimes Y \to Y$$

are both identities.

The object Y and morphisms e, i are defined uniquely up to a canonical isomorphism. We will denote this object Y by X^* and call it the *right dual* to X.

We similarly introduce the notion of a *right rigid object* and denote by ${}^*X$ its *left dual* (so $({}^*X)^* = X$).

The monoidal category C is called a *rigid* one if all its objects are left and right rigid.

(1.1.4) **Example**. Let G be a group. Then the category Rep(G) of finite dimensional representations of G is rigid. In this case $X^* = {}^*X$ is the dual representation.

(1.1.5) It is easy to see that in a rigid monoidal category we have a canonical isomorphism $(X \otimes Y)^* = Y^* \otimes X^*$. This implies that the functor $X \mapsto X^{**}$ has a canonical structure of a monoidal functor, i.e. it is equipped with a canonical functorial isomorphism $(X \otimes Y)^{**} \approx X^{**} \otimes Y^{**}$.

(1.1.6) Another important classical notion connected with monoids is the notion of a ring. In case of monoidal categories this corresponds to the additive monoidal category C, such that the multiplication functor F is biadditive . Example of such category is the category of representations of a group (see example (1.1.4)), or of representations of a quantum group, which we will discuss in lecture 2.

Note that we may have an abelian monoidal category in which all objects are rigid. This has no analogues in the classical case.

(1.1.7) Let C be an additive monoidal category. We can define a notion of a module category over C, which generalizes the notion of a module over a ring. Namely, such category consists of an additive category $\mathcal{M}$, a biadditive multiplication functor $H : C \times \mathcal{M} \to \mathcal{M}$ and an associativity constraint $b_{X,Y,M} : (X \otimes Y) \otimes M \tilde{\to} X \otimes (Y \otimes M)$, satisfying the pentagon identity.

1.2. Tensor categories.

(1.2.1) Let C be a monoidal category. A *symmetry constraint* S is a collection of isomorphisms $S = \{S_{XY} : X \otimes Y \tilde{\to} Y \otimes X\}$ satisfying two hexagon axioms (H1) and (H2):

$$(H1) \qquad \begin{array}{ccccc} X \otimes (Y \otimes Z) & \to & (X \otimes Y) \otimes Z & \xrightarrow{S^+} & Z \otimes (X \otimes Y) \\ \downarrow \mathrm{Id} \otimes S^+ & & & & \downarrow \\ X \otimes (Z \otimes Y) & \to & (X \otimes Z) \otimes Y & \xrightarrow{S^+ \otimes \mathrm{Id}} & (Z \otimes X) \otimes Y \end{array}$$

where $S^+ = S$.

Axiom $(H2)$ is obtained from $(H1)$ by replacing family S^+ with the family S^- defined by $S^-_{XY} = (S_{YX})^{-1}$.

(1.2.2) Until about 10 years ago it was always assumed that $S^- = S^+$, i.e. $S_{YX} \cdot S_{XY} =$ id (now such categories are called *symmetric* ones). For example the category Rep(G) of representations of a group G is symmetric.

On the other hand, the category of representations of a quantum group satisfies a symmetry constraint but is not symmetric.

Definition. A *tensor category* is an abelian rigid monoidal category C equipped with a symmetry constraint S.

(1.2.3) A symmetry constraint S defines for any pair of objects $X, Y \in C$ a functorial automorphism $S_{YX} \circ S_{XY}$ of the object $X \otimes Y$. If the category C is not symmetric, this automorphism $S_{YX} \circ S_{XY}$ is nontrivial. In many interesting examples there is an additional structure which rigidifies this isomorphism.

Definition. A *balancing* on a tensor category (C, S) is an automorphism t of the identity functor on C such that

$$S_{YX} \circ S_{XY} = t_{X \otimes Y} \circ t_X^{-1} \circ t_Y^{-1}$$

and

$$(t_{X^*}) = (t_X)^*.$$

A tensor category C equipped with a balancing t is called a *balanced tensor category.*

(1.2.4) For a balanced tensor category we can identify the two dual objects X^* and ${}^*X$. Namely, for any tensor category C we have a canonical morphism $\alpha_X : X \to X^{**}$, given by the composition

$$\alpha_X : X \to X \otimes \mathbb{1} \to X \otimes X^* \otimes X^{**} \to X^* \otimes X \otimes X^{**} \to \mathbb{1} \otimes X^{**} \to X^{**} .$$

This morphism is not a morphism of monoidal functors. However, if we consider morphism $\beta_X = \alpha_X \circ t_X^{-1} : X \to X^{**}$ then β_X is an isomorphism of monoidal functors Id and $X \mapsto X^{**}$ from C to C.

Using this isomorphism we will identify X with X^{**} and hence ${}^*X$ with X^*.

1.3. Invariants of knots.

(1.3.1) Let C be a tensor category, X and Y two objects in C. By permuting X with Y we turn $X \otimes Y$ into $Y \otimes X$. These two objects are isomorphic, but we have to choose between two natural isomorphisms $S^+ = S_{XY}$ and $S^- = S_{YX}^{-1}$.

Informally speaking, we may say that an isomorphism between the product $X \otimes Y$ and the permuted product $Y \otimes X$ depends on how X and Y moved pass each other: if X passed "over" Y we will use S^+, if X passed "under" Y we will use S^-.

If we have n objects $X_1, \ldots, X_n$, then for any permutation σ of indices $\{1, \ldots, n\}$ we can consider the object $Z_\sigma = X_{\sigma(1)} \otimes \ldots \otimes X_{\sigma(n)}$. All these objects are isomorphic, but the choice of an isomorphism depends on the way the objects X_i pass over or under each other.

(1.3.2) There is a geometric notion which generalizes the notion of permutation and takes into account the over/under relation between permuted objects. This is a notion of *Artin's braids.*

Let us recall that a *braid* b acting on a set $I = \{1, \dots, n\}$ and realizing the permutation σ is a continuous family of imbeddings $b_u : I \to \mathbb{C}$, which starts with the identity imbedding at $u = 0$ and ends with the imbedding defined by σ at $u = 1$. The set B_n of isotopy classes of such braids has a natural group structure. It is called the *braid group* of order n.

Claim. Let $b \in B_n$ be a braid acting on the set $\{1, \dots, n\}$ and realizing the permutation σ. Then for objects $X_1, \dots, X_n \in C$ the braid b induces a well defined isomorphism $\gamma_b : Z_e \widetilde{\to} Z_\sigma$. Moreover, $\gamma_{b_1 b_2} = \gamma_{b_1} \circ \gamma_{b_2}$.

Thus, starting from a purely algebraic object — a tensor category — we can construct a representation of such a geometric object as the braid group B_n.

(1.3.3) Suppose we have fixed a balanced tensor category (C, S, t). Then in a way similar to (1.3.2) we can construct an algebraic representation of a geometric object.

Namely, let L be a *link*, i.e. a collection of knotted oriented circles in $\mathbb{R}^3$. Suppose that to each circle α we have assigned a *colouring*, which is an object $X_\alpha \in C$. We want to define a weight $w = w(L, \{X_\alpha\})$.

Let us choose a generic oriented plane M in $\mathbb{R}^3$ and a generic linear function y on M. We will consider the projection of our link L on the plane M and study its intersection with horizontal lines.

For any horizontal straight line λ (given by the equation $y = \lambda$) which intersects the projection of L in general position we consider an object $X_\lambda \in C$, given by $X_\lambda = W_{\nu_1} \otimes W_{\nu_2} \otimes \dots \otimes W_{\nu_k}$. Here $\nu_1 \dots \nu_k$ are points of intersection of the line λ with the projection of L (the order of these points is determined by the orientation of λ defined by the equation $y = \lambda$ and the orientation of M). For every point ν_i the factor W_{ν_i} equals either X_α or X_α^*, where α is the circle which passes through ν_i, and we choose X_α if the circle goes up at this point and X_α^* if it goes down.

(1.3.4) Let us see what will happen with the object X_λ when we move the level λ from $-\infty$ to ∞. It is clear that locally we will only have movements of the following four types:

I. Two neighboring objects $X = W_{\nu_i}$ and $Y = W_{\nu_{i+1}}$ interchange, so that X passes over Y.

II. Two neighboring objects X and Y interchange, so that X passes under Y.

III. Two neighboring objects X and X^* collide and dissapear.

IV. Two neighboring objects X^* and X are born out of thin air.

In other words we can choose a sequence $\lambda_1 < \lambda_2 < \dots < \lambda_N$ such that

1) $\lambda_1 \ll 0$, so $X_{\lambda_1} = \mathbb{1}$;

2) $\lambda_N \gg 0$, so $X_{\lambda_N} = \mathbb{1}$;

3) In passage from λ_i to λ_{i+1} only one of the four simple moves above appears.

Let us define morphisms $m_i : X_{\lambda_i} \to X_{\lambda_{i+1}}$ by using
$S^+ : X \otimes Y \to Y \otimes X$ in case I
$S^- : X \otimes Y \to Y \otimes X$ in case II
$e : X \otimes X^* \to \mathbb{1}$ in case III and
$i : \mathbb{1} \to X^* \otimes X$ in case IV.

Now for any $i < j$ we define a mophism $m_{ij} : X_{\lambda_i} \to X_{\lambda_j}$ as the composition $m_{ij} = m_{j-1} \circ \ldots \circ m_i$.

In particular, we have defined a morphism $m_{1N} : \mathbb{1} \to \mathbb{1}$.

If we assume that $\mathrm{End}(\mathbb{1}) = \mathbb{C}$, then this morphism m_{1N} is a number. This number, which we denote by $w(L, \{X_\alpha\}; M, y)$ is the weight we wanted to describe.

(1.3.5) In this definition the weight w depends on the choice of an oriented plane M and an ordinate y on M. In fact the axioms of a balanced tensor category imply that, to a large extent, this weight only depends on L and the colourings X_α.

In order to get invariants independent of M and y we have to pass to a *framed* link. So let us consider a framed link L, i.e. a link L together with a field f of nonzero normal vectors on L. First assume that our framed link L is in a good position, so that we can choose an oriented plane M in such a way, that the frame f on L is everywhere positively transversal to M. Then one can show that the wight $w(L, \{X_\alpha\}; M, y)$ does not depend on M and y.

Now it is easy to see that any framed link (L, f) is isotopic to a framed link (L', f') in a good position. Axioms of a rigid balanced tensor category then imply that the resulting weight $w(L', f')$ depends only on the isotopy class of the coloured framed link $(L, f, \{X_\alpha\})$.

Thus we see, that a balanced tensor category gives a way to produce invariants of framed links.

(1.3.6) **Remark.** This construction in fact produces a representation of an algebraic object – the category of *tangles.*

(1.3.7) How to produce invariants of links independent of the colouring ? One way to do it is to assign the same colouring to all circles. However physical interpretation of our picture suggests a better way to do it. Namely, it suggests to consider the sum of the weights over all possible colourings.

We will see in Lecture 3 that this procedure works quite well when the category C has finite number of simple objects X_α. Summing over all possible colourings of the circles of our link L by these simple objects produces an invariant of framed links $w(L, f)$, which only depends on our balanced category C.

Lecture 2. Quantum Groups.

Quantum groups were introduced independently by Drinfeld and Jimbo around 1985. The term "quantum group" was coined by Drinfeld who also proposed their theory. Quantum groups soon became very popular and were further studied by many people. I will mostly follow Lusztig and Manin.

I will discuss quantum groups from the point of view of tensor categories.

2.1. Definition of quantum groups.

(2.1.1) Let G be a finite group, $A = \mathbb{C}[G]$ the algebra of functions on G with respect to pointwise multiplication. This algebra has the following structures :

(i) A is an associative algebra with the unit element 1. Thus we have two morphisms

$$\begin{array}{lll} m & : & A \otimes A \to A \quad \text{(multiplication)} \quad \text{and} \\ \eta & : & \mathbb{C} \to A \qquad\quad\ \ \text{(unit)}. \end{array}$$

(ii) The multiplication map $G \times G \to G$ defines a morphism $\Delta : A \to A \otimes A$ (comultiplication) by $\Delta f(x,y) = f(xy)$. The imbedding of the identity $\{e\} \hookrightarrow G$ defines a morphism $\varepsilon : A \to \mathbb{C}$ (counit) by $\varepsilon(f) = f(e)$.

(2.1.2) **Definition.** A *bialgebra* over $\mathbb{C}$ is a linear space A equipped with the operations

$$\begin{array}{llllll} m & : & A \otimes A & \to & A & \text{(multiplication)} \\ \Delta & : & A & \to & A \otimes A & \text{(comultiplication)} \\ \eta & : & \mathbb{C} & \to & A & \text{(unit)} \\ \varepsilon & : & A & \to & \mathbb{C} & \text{(counit)} \end{array}$$

satisfying the following axioms :

$H1.1.$ m is associative;
$H1.2.$ η is a unit with respect to m;

$H2.1.$ Δ is coassociative ;
$H2.2.$ ε is a counit with respect to Δ;

H3 (connecting axioms):

$H3.1.$ Δ is a morphism of algebras;
$H3.2.$ ε is a morphism of algebras;
$H3.3.$ η is a morphism of coalgebras .

These three axioms can be expressed as the commutativity of some diagrams, constructed in terms of $m, \Delta, \eta, \varepsilon$. For example, the axiom $H3.1$ is written in a symmetric form

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Id}} & A \\ \uparrow m & & \downarrow \Delta \\ A \otimes A & & A \otimes A \\ \downarrow \Delta\otimes\Delta & & \uparrow m\otimes m \\ A \otimes A \otimes A \otimes A & \xrightarrow{S_{2,3}} & A \otimes A \otimes A \otimes A \,. \end{array}$$

It is equivalent to the statement that m is a morphism of coalgebras.

(2.1.3) **Definition.** A bialgebra A is called a *Hopf algebra* if there exists an *antipode morphism* $inv : A \to A$ such that the two diagrams – for $i = inv \otimes \mathrm{id}$ and for $i = \mathrm{id} \otimes inv$ – are commutative:

$$\begin{array}{ccccc} A \otimes A & & \xrightarrow{i} & & A \otimes A \\ \uparrow \Delta & & & & \downarrow m \\ A & \xrightarrow{\varepsilon} & \mathbb{C} & \xrightarrow{\eta} & A \end{array}$$

It is easy to check that the antipode morphism, if exists, is uniquely defined. It reverses m and Δ.

(2.1.4) **Example.** Let A be a finite dimensional commutative semisimple algebra over $\mathbb{C}$. Then it can be realized as $A = \mathbb{C}[G]$, where G is the finite set $G = \operatorname{Spec} A$. A comultiplication morphism $\Delta : A \to A \otimes A$ defines a multiplication map $G \times G \to G$. If A is a bialgebra, this map defines on G the structure of a monoid with the unit given by ε. If A is a Hopf algebra, then G is a group.

(2.1.5) More generally, let A be a commutative finitely generated algebra over $\mathbb{C}$. For simplicity assume that it does not have nilpotent elements. Then it can be realized as the algebra of regular functions on an algebraic variety G. To define on A the structure of a bialgebra is the same as to equip G with the structure of an algebraic monoid. The condition that A is a Hopf algebra means that G is an affine algebraic group.

(2.1.6) Let $(A, m, \Delta, \varepsilon, \eta)$ be a finite dimensional Hopf algebra.

Consider the dual vector space A^* and the adjoint operators $m^*, \Delta^*, \eta^*, \varepsilon^*$. Then $(A^*, \Delta^*, m^*, \eta^*, \varepsilon^*)$ is also a Hopf algebra. It is called the *dual Hopf algebra* to A.

For infinite dimensional Hopf algebras this definition does not work since the comultiplication morphism m^* is not well-defined. The situation, however, can often be rectified by considering an appropriate completion of $A^* \otimes A^*$, or passing to a subalgebra $U \subset A^*$ on which m^* is defined.

(2.1.7) In all examples of Hopf algebras A we have considered, A is either commutative (e.g. $A = \mathbb{C}[G]$), or cocommutative (e.g. $A = \mathbb{C}[G]^*$). A natural question is whether there exist *natural* examples of Hopf algebras, which are neither commutative nor cocommutative.

What Drinfeld and Jimbo have discovered is the family of such Hopf algebras (Drinfeld called them quantum groups). We will formulate the result of Drinfeld and Jimbo in a form closer to Manin's description.

Statement. *Let G be a simple algebraic group over $\mathbb{C}$, $A = \mathbb{C}[G]$ the Hopf algebra of regular functions on G. Then there exists a nontrivial family of Hopf algebras (quantum groups) A_q parametrized by $q \in \mathbb{C}^*$ such that $A_1 = A$.*

(2.1.8) Let us describe in detail the case of $G = SL(2, \mathbb{C})$. In this case the algebra A is generated by four generators $a,\ b,\ c,\ d$ which commute and satisfy the relation $ad - bc = 1$.

In order to describe the comultiplication $\Delta : A \to A \otimes A$ we introduce a matrix

$$Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, A) .$$

Using natural imbeddings $i', i'' : A \to A \otimes A$, $(i'(x) = x \otimes 1,\ i''(x) = 1 \otimes x)$, we can define morphism $\Delta : A \to A \otimes A$ by the formula

$$(*) \qquad \Delta(Y) = i'(Y) \cdot i''(Y) ,$$

which is an equality in $Mat(2, A \otimes A)$. Condition $(*)$ is a shorthand for equalities

$$\Delta a = a \otimes a + b \otimes c$$
$$\Delta b = a \otimes b + b \otimes d \qquad \text{and so on.}$$

Now we can describe the quantum group $SL_q(2)$. It is given by an algebra A_q, generated by elements (a, b, c, d), satisfying the following relations

$$(**) \qquad \begin{array}{ll} ab = q^{-1} ba & ac = q^{-1} ca \\ cd = q^{-1} dc & bd = q^{-1} db \\ bc = cb & ad - da = (q^{-1} - q) bc \end{array}$$

$$ad - q^{-1} bc = 1 .$$

Comultiplication Δ is given by the same condition $(*)$ as above.

In Manin's Montreal notes there is a beautiful proof which shows that A_q is a Hopf algebra and explains the origin of conditions $(**)$.

2.2. Representations of quantum groups.

(2.2.1) Let ρ be a representation of a finite group G in a vector space V. It is possible to define ρ in terms of the coalgebra $A = \mathbb{C}[G]$ as a morphism $\rho : V \to A \otimes V$ which satisfies the following axioms (roughly speaking the axioms of a G-module with reversed arrows in the commutative diagrams which define it):

R1) ρ is coassociative, i.e. the two natural morphisms $V \to A \otimes A \otimes V$ coincide,
R2) the counit ε acts on V as identity.
This example is a model for the following general notion.

(2.2.2) **Definition.** Let A be a coalgebra. Then a *comodule* over A is a morphism $\rho : V \to A \otimes V$, satisfying axioms $R1$, $R2$.

(2.2.3) If V is a comodule over A, then any element $\varphi \in A^*$ defines an operator $\rho(\varphi) : V \to V$ as the through map $V \xrightarrow{\rho} A \otimes V \xrightarrow{\varphi} V$.
The axioms $R1$, $R2$ are equivalent to the condition that V is an A^*-module.

When A is finite dimensional this functor A-comod $\to A^*$-mod gives an equivalence between the categories of A-comodules and A^*-modules. If A is infinite dimensional this functor is fully faithful, but is not an equivalence.

(2.2.4) **Example.** Let $V = \mathbb{C}^n$. Then a morphism $\rho : \mathbb{C}^n \to A \otimes \mathbb{C}^n$ defines a matrix $Y \in \mathrm{Mat}(n, A)$ by $\rho(e_i) = \Sigma y_{ij} \otimes e_j$. The condition that ρ defines a comodule structure on $\mathbb{C}^n$ is equivalent to the condition that the matrix Y is *multiplicative*, i.e. $\Delta(Y) = i'(Y) \otimes i''(Y)$ (cf. 2.1.8).

(2.2.5) It is natural to define a *finite quantum group* as a finite dimensional Hopf algebra A. Let us also define an *algebraic quantum group* as a Hopf algebra A such that

(i) A is a finitely generated algebra,
(ii) A is a union of finite dimensional A-comodules.

By analogy with the classical situation we can formulate two problems:

I) How to describe finite quantum groups.
II) How to describe algebraic quantum groups.

(2.2.6) Let us discuss the problem II in a special case of quantum groups G_q which are deformations of a classical simple group G. It turns out that for groups G different from $SL(2)$ it is difficult to give an explicit description of the algebra A_q. The reason for this is that the algebra $A = \mathbb{C}[G]$ is already quite complicated. (Do you know the description of this algebra for the group G of type F_4 ?)

In order to describe this algebra one usually passes to the Lie algebra $\mathfrak{g} = Lie(G)$ and to the envelopping algebra $U(\mathfrak{g})$. This algebra $U(\mathfrak{g})$ lies in A^* and inherits the structure of a Hopf algebra from A^*. While A^* is, clearly, too big, the algebra $U(\mathfrak{g})$ is quite manageable.

Similarly in the quantum case one usually describes instead of the complicated Hopf algebra A_q a simpler Hopf algebra $U_q \subset A_q^*$. Then, in terms of the algebra U_q, it is usually not difficult to reconstruct the Hopf algebra A_q.

(2.2.7) **Example.** $G_q = SL_q(2)$.

Let $I \subset A_q$ be an ideal generated by b and c, $S = A_q/I = \mathbb{C}[a,d]/(ad-1)$ (see (2.1.8)). Then I is a Hopf ideal in A_q and S is a Hopf algebra isomorphic to the Hopf algebra of regular functions on the algebraic group $H = \mathbb{C}^*$. Informally, this means that our quantum group G_q contains H as a subgroup.

In particular, the group H acts on the algebra A_q from the left and from the right.

Consider elements $E, F, K \in A_q^*$ defined as follows. Identify A/I^2 with $S \oplus S\,b \oplus S\,c$ and define $E, F, K : A/I^2 \to \mathbb{C}$ by

$$K(P + Qb + Rc) = P(q)$$
$$E(P + Qb + Rc) = Q(1)$$
$$F(P + Qb + Rc) = R(1)\ .$$

Theorem. *Let U_q be a subalgebra of A_q^* generated by $E, F, K^{\pm 1}$. Then U_q is a Hopf algebra, given by the following relations*

$$KE = q^2 EK \qquad KF = q^{-2} FK$$

$$[E,F]=\frac{K-K^{-1}}{q-q^{-1}}$$

$$(L)\qquad \begin{aligned}\Delta K &= K\otimes K\\ \Delta E &= E\otimes 1+K\otimes E\\ \Delta F &= F\otimes K^{-1}+1\otimes F\ .\end{aligned}$$

This is the Lusztig's form of generators for U_q.

(2.2.8) For the general simple group G Lusztig has described the algebra U_q as a Hopf algebra, generated by generators $E_i, F_i, K_i^{\pm 1}$ satisfying some relations similar to (L).

Representations of the quantum group G_q can be realized as some special U_q-modules.

2.3. Representations of a Hopf algebra as a monoidal category.

(2.3.1) Let (ρ, V), (σ, E) be two comodules over a Hopf algebra A. Then using the multiplication morphism $m : A\otimes A\to A$ we can define an A-comodule structure on the space $V\otimes E$ by setting

$$V\otimes E\xrightarrow{\rho\otimes\sigma}(V\otimes A)\otimes(E\otimes A)\xrightarrow{S_{2,3}}V\otimes E\otimes A\otimes A\xrightarrow{m}V\otimes E\otimes A\ .$$

Let $\mathrm{Rep}(A)$ be the category of finite dimensional A-comodules. Then the tensor product described above defines on $\mathrm{Rep}(A)$ the structure of a monoidal category. Using the antipode one can show that this category is rigid.

The forgetful functor $(\rho, V)\mapsto V$ defines a monoidal functor $H : \mathrm{Rep}(A)\to \mathrm{Vec}$, where Vec is the category of finite dimensional vector spaces.

(2.3.2) The following theorem, due to Majid, is philosophically significant, since it clarifies the relation between Hopf algebras and tensor categories.

Definition. Let R be a monoidal category. A *fiber functor* is a functor $H : R\to \mathrm{Vec}$, together with a functorial isomorphism

$$\eta_{X,Y} : H(X\otimes Y)\xrightarrow{\sim} H(X)\otimes H(Y)\ ,$$

which is compatible with the associativity constraints in the categories R and Vec.

(2.3.3) **Theorem.** *Let R be an abelian rigid monoidal category and $H : R\to \mathrm{Vec}$ a fiber functor. Then there exists a Hopf algebra A and an equivalence of monoidal categories $R\approx \mathrm{Rep}(A)$ under which H becomes the forgetful functor. The Hopf algebra A is uniquely defined up to a canonical isomorphism.*

(2.3.4) Let R be an abelian rigid monoidal category. Suppose it has a fiber functor $H : R\to \mathrm{Vec}$. Every such functor leads to a Hopf algebra $A = A(H)$. For nonisomorphic functors these Hopf algebras are not isomorphic but it is clear that one should consider them as equivalent since they describe essentially the same algebraic object.

The corresponding notion of equivalence for Hopf algebras was introduced by Drinfeld. Namely, consider an element $Q \in (A \otimes A)^*$. Then using the left and right $A \otimes A$-comodule structures on $A \otimes A$ we define the operators $\rho_L(Q)$ and $\rho_R(Q)$ on $A \otimes A$. Note that these operators only depend on the coalgebra structure on A.

Now, consider a new algebra structure on A given by the multiplication operator $m_Q : A \otimes A \to A$, where $m_Q = m \circ \rho_L(Q) \circ \rho_R(Q)^{-1} : A \otimes A \to A$. (In order to define the multiplication m_Q we have to assume that the operator $\rho_R(Q)$ is invertible.)

It is easy to see that m_Q is always a morphism of coalgebras. The associativity condition imposes rather complicated restrictions on Q. (Drinfeld avoids these complications by introducing the notion of a quasi-Hopf algebra which is a generalization of the notion of Hopf algebra.) Drinfeld calls the new Hopf algebra $A_Q = (A, m_Q, \Delta)$ *gauge equivalent* to the original Hopf algebra $A = (A, m, \Delta)$.

Since, as a coalgebra, $A_Q = A$, we have a natural (identity) equivalence of categories $I : \mathrm{Rep}(A) \simeq \mathrm{Rep}(A_Q)$. This equivalence can be extended to an equivalence of monoidal categories; namely, given two A-comodules (ρ, V) and (σ, E) we define an isomorphism $\alpha_{\rho,J\sigma} : I(\rho) \otimes I(\sigma) \widetilde{\to} I(\rho \otimes \sigma)$ using an operator $(\rho \otimes \sigma)(Q) \in \mathrm{Aut}(V \otimes E)$ (both A_Q-comodules $I(\rho) \otimes I(\sigma)$ and $I(\rho \otimes \sigma)$ are realized in the same vector space $V \otimes E$ and the automorphism $(\rho \otimes \sigma)(Q)$ defines an isomorphism between them).

(2.3.5) After the original Drinfeld-Jimbo construction of a 1-parametric family of deformations of $A = \mathbb{C}[G]$ several people noticed that for every simple group G there actually exists a family of deformations of the Hopf algebra A, which depends on a large number of parameters and the Hopf algebras in the family are nonisomorphic. (For example for $G = SL(n, \mathbb{C})$ this family of Hopf algebras depends on $\approx n^2/2$ parameters.) Later Drinfeld showed that every deformation of A only depends on one parameter (say, the original Drinfeld-Jimbo's one) if considered up to a gauge equivalence.

(2.3.6) Given a Hopf algebra A, we can consider a new Hopf algebra A° with the opposite multiplication m° and the same comultiplication Δ. When A is a deformation of the algebra $\mathbb{C}[G]$, Drinfeld's results imply that this new Hopf algebra A° is gauge equivalent to A.

Let $R \in (A \otimes A)^*$ be an element which defines this equivalence (Drinfeld calls it the *universal R-matrix*). It is easy to see that R defines a symmetry constraint $S_{XY} : X \otimes Y \widetilde{\to} Y \otimes X$ on the category $\mathrm{Rep}(A)$. Thus, $\mathrm{Rep}(A)$ is a tensor category.

Actually, Drinfeld has also constructed a balancing on this category. So $\mathrm{Rep}(A)$ is a balanced tensor category.

(2.3.7) Another natural possibility suggested by Theorem 2.3.3 is that there might exist an abelian rigid monoidal category R which does not have any fiber functor. Then R is not directly related to any Hopf algebra, though intuitively it represents a mathematical object of the same nature. We will see natural examples of such categories in 2.4.

(2.3.8) Usually the category $R = \mathrm{Rep}(A)$ is studied from the dual point of view. Namely, one fixes a Hopf subalgebra $U \subset A^*$ and considers the category R of finite dimensional U-modules with tensor product given by Δ, i.e.

$$U \otimes (V \otimes E) \xrightarrow{\Delta} U \otimes U \otimes (V \otimes E) \to (U \otimes V) \otimes (U \otimes E) \to V \otimes E\,.$$

Drinfeld, Jimbo and Lusztig only work with this dual picture, and only study the Hopf algebra U_q with practically no reference to the algebra A_q. They also use slightly different subalgebras U inside the algebra A_q^*.

(2.3.9) Quantum groups U_q studied by Lusztig are deformations of the algebra $U_1 = U(\mathfrak{g})$, which is the envelopping algebra of a simple Lie algebra $\mathfrak{g}$. Lusztig showed that any finite dimensional representation V of $U(\mathfrak{g})$ admits a deformation V_q which is a representation of the algebra U_q in the same vector space V.

For a generic q this construction allows us to describe all U_q-modules. When q is a root of 1 the situation is much more interesting and complicated. For example, in this case the category $\mathrm{Rep}(G_q)$ as well as the category of U_q-modules are not semisimple.

2.4. Finite quantum groups.

(2.4.1) Let G be a simple algebraic group $A = \mathbb{C}[G]$. Fix a prime number l and consider a quantum group A_q, where $q = \sqrt[l]{1}$. Then there exists a natural Frobenius morphism of Hopf algebras $Fr : A \to A_q$, which can be interpreted as a homomorphism of quantum groups $Fr_q^* : G_q \to G$.

(2.4.2) **Example**. Consider $G = SL(2)$. Then Fr is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^l & b^l \\ c^l & d^l\,. \end{pmatrix}$$

(2.4.3) Let H_q be the kernel of the homomorphism of quantum groups $Fr^* : G_q \to G$. This kernel is a finite quantum group, which is given by the finite dimensional Hopf algebra $H_q = A_q/J$, where J is the ideal generated by $Fr\big(\mathrm{Ker}(\varepsilon : A \to \mathbb{C})\big)$. For example, if $G = SL(2)$ the ideal J is generated by $\{b^l, c^l, a^l - 1, d^l - 1\}$.

The quantum group H_q is the quantum analogue of the finite group $G(\mathbb{F}_l)$. This analogy is explored in detail in Lusztig's works.

(2.4.4) The category $R_q = \mathrm{Rep}(H_q)$ is usually not semisimple. It turns out that one can modify it (replace by an appropriate quotient category) so that the resulting category $\overline{R}_q$ is a semisimple tensor category with a finite number of simple objects. Let us describe how to do it in case of $SL(2)$.

(2.4.5) The group $G = SL(2, \mathbb{C})$ has a series of irreducible representations V_i, $i = 0, 1, 2, \dots$, with $\dim V_i = i + 1$. For any q we will denote also by V_i the deformed representations of the quantum group G_q.

Let $q = \sqrt[l]{1}$. Let V be an H_q-comodule. Then V is an A_q-comodule, and using formulas in (2.2.7) and (2.2.3) we can define operators $E, K, F : V \to V$.

We say that the H_q-comodule V is *negligible* if it is free as $\mathbb{C}[E]/(E^l)$-module.

Let us define the category $\overline{R}_q$ as the quotient of the category R_q modulo all negligible objects. One can show that this notion is well-defined and the resulting category $\overline{R}_q$ is an abelian semisimple category with isomorphism classes of simple objects given by $V_0, \ldots, V_{l-2}$ (these modules are simple objects in R_q). Since the tensor product of a negligible object by any other object is negligible, $\overline{R}_q$ inherits the structure of a balanced tensor category from the category R_q.

(2.4.6) Thus, for $q = \sqrt[l]{1}$ we have described a semisimple balanced tensor category with a finite number of simple objects. One can construct similar categories starting with any simple group G.

One can check that these categories do not have fiber functors, so they do not admit a direct description in terms of Hopf algebras as in 2.3; we have, however, described them using Hopf algebras A_q and H_q.

These categories are the sources of nontrivial invariants of knots described in lecture 1 and of topological field theories which we are going to discuss in lecture 3.

Lecture 3. Topological (quantum) field theories.

3.1. Topological field theories (TFT).

TFT immerged in physics in study of conformal field theories. They were first explicitly studied by E. Witten. The mathematical framework for TFT is described in Atiyah's paper on TFT.

(3.1.1) Let us fix terminology: a *manifold* is a compact oriented manifold M with boundary. Its boundary ∂M will be considered with the induced orientation. We say that M is closed if $\partial M = \emptyset$.

We use notation M^* for the manifold M with opposite orientation.

We will describe TFT of dimension $(d, 1)$ (in physical interpretation we have 1 time variable and d space variables).

Usually we will use the following mnemonic rule: we denote by N the $(d+1)$-manifolds (i.e manifolds of dimension $d+1$), by M d-manifolds, by L $(d-1)$-manifolds and so on.

(3.1.2) We are going to describe a topological field theory W of dimension $(d, 1)$. This will be done on several levels.

Definition of TFT on level 0. A TFT W is a multiplicative invariant of closed $(d+1)$-manifolds.

In other words W assigns to every closed $(d+1)$-manifold N a number $w(N)$ such that $w(N \cup N') = w(N) \cdot w(N')$.

(3.1.3) **Definition of TFT on level 1.**

A TFT W assigns to every closed d-manifold M a finite dimensional vector space $W(M)$ and to every bordism N between two d-manifolds M_1 and M_2 it assigns an operator $w(N) : W(M_1) \to W(M_2)$.

(3.1.4) In definition (3.1.3) we assume that the correspondence $M \mapsto W(M)$ is functorial with respect to diffeomorphisms of d-manifolds.

(3.1.5) We also assume that this correspondence is multiplicative. This meams that in addition to the functor $M \mapsto W(M)$ we are given a functorial isomorphism $\gamma_{M,M'}$: $W(M \cup M') \tilde{\rightarrow} W(M) \otimes W(M')$.

(3.1.6) By definition a bordism between two closed d-manifolds M_1 and M_2 is a $(d+1)$-manifold N equipped with an isomorphism $\partial N \approx M_1^* \cup M_2$ (here M_1^* is the manifold M_1 with the opposite orientation).

(3.1.7) The data, described in (3.1.3), i.e. the functor W, the system of functorial isomorphisms $\gamma_{M,M'}$ and the system of operators $w(N)$ should satisfy the following requirements.

(i) (W, γ) is a monoidal functor compatible with the natural symmetry constraint. In other words $W(\emptyset) = \mathbb{C}$ and the system of isomorphisms γ is compatible with the natural associativity constraint, with the unit object and with the natural symmetry constraint. For example, the last condition just means that the natural diffeomorphism $M \cup M' \approx M' \cup M$ will correspond to the standard isomorphism $W(M) \otimes W(M')$ with $W(M') \otimes W(M)$.

(ii) *Functoriality.* Diffeomorphic bordisms give the same operator.

(iii) *Composition.* Suppose we are given three closed d-manifolds M_1, M_2, M_3 and two bordisms N' between M_1 and M_2 and N'' between M_2 and M_3. Then we can glue them and get a bordism N between M_1 and M_3 (notation $N = N'' \circ N'$). We require that
$$w(N) = w(N'') \circ w(N') \ .$$

(iv) *Cylinder.* Let $C(M) = [0,1] \times M$ be the cylinder bordism between M and M. Then $w\big(C(M)\big) = \mathrm{Id}_{W(M)}$.

(v) *Multiplicativity.* If N is a bordism between M_1 and M_2 and N' a bordism between M_1' and M_2', then
$$w(N \cup N') = w(N) \otimes w(N').$$

(3.1.8) **Comment 1.** From axioms of TFT follows the following homotopy invariance property.

Lemma. Let $\varphi_t : M \to M'$ be a smooth family of diffeomorphisms. Then the morphism $W(\varphi_t) : W(M) \to W(M')$ locally does not depend on t.

This shows, that TFT W defines a representation of the mapping class group $Cl(M) = \mathrm{Diff}(M)/\,\mathrm{Diff}^0(M)$ in the vector space $W(M)$.

(3.1.9.) **Comment 2.** TFT W defines an invariant of closed $(d+1)$-manifolds. Indeed, every such manifold N can be considered as a bordism between $M_1 = M_2 = \emptyset$. Hence $w(N) \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$. This shows the connection of descriptions of level 1 and level 0.

Note that usually TFT can be uniquely reconstructed from this invariant of $(d+1)$-manifolds.

(3.1.10) **Comment 3.** Let N be a $(d+1)$-manifold and $M = \partial N$. We can interpret N as a bordism between empty manifold $\emptyset$ and M. Then the operator $w(N) \in \mathrm{Hom}(\mathbb{C}, W(M))$ is nothing else as a vector $w(N) \in W(M)$. Similarly, if $\partial N = M^*$, we can interpret $w(N)$ as a functional on $W(M)$.

(3.1.11) **Comment 4.** Let N be a closed $(d+1)$-manifold. Suppose we have cut it into two pieces N_1 and N_2 with the common boundary M. Then $w(N_1)$ is a vector in $W(M)$ and $w(N_2)$ is a functional on $W(M)$. The composition property implies that

$$\langle w(N_2),\ w(N_1)\rangle = w(N)\ .$$

This shows, that for a TFT W the corresponding invariant of $(d+1)$-manifolds has local character in a sense, that the number $w(N)$ can be computed by cutting N in pieces. The important and highly nontrivial feature of topological field theories is that the answer does not depend on the nature of the cut.

(3.1.12) **Comment 5.** For a closed d-manifold M we can interpret the cylinder $N = C(M)$ as a bordism between $M \cup M^*$ and the empty manifold $\emptyset$. This defines a canonical pairing $w(N) : W(M) \otimes W(M^*) \to \mathbb{C}$. From axioms of TFT follows, that this pairing is nondegenerate, so it defines a canonical isomorphism $W(M^*) \approx W(M)^*$.

Similarly we have a canonical morphism $w(N) : \mathbb{C} \to W(M) \otimes W(M^*)$, which induces the same isomorphism $W(M^*) \simeq W(M)^*$.

3.2. Fusion Algebra.

We will be mostly interested in case $d = 2$. But first let us look at the case $d = 1$.

(3.2.1) Fix a TFT W of dimension $(1, 1)$.

Consider an (oriented) unit circle S and set $A = W(S)$. This is a well defined vector space, since the group $\mathrm{Diff}^+(S)$ of orientation preserving diffeomorphisms of S acts trivially on A.

We have a canonical (up to homotopy) diffeomorphism $\theta : S \to S^*$. It defines an isomorphism $\theta : W(S) \to W(S^*) \simeq W(S)^*$, i.e. a bilinear form θ on A. It is easy to see that θ is symmetric and nondegenerate.

(3.2.2) Let N be a 2-sphere with 3 disjoint discs removed (it is usually called *pants*). We consider N as a bordism between $S \cup S$ and S. Then the operator $w(N)$ defines multiplication

$$m : A \otimes A \to A\ .$$

Axioms of TFT imply that m is associative (this follows from the geometric fact that bordisms $N \circ (N \circ N)$ and $(N \circ N) \circ N$ are diffeomorphic) .

It turns out that it is also commutative. In order to see this it is enough to consider a rotation φ of pants through $180°$, so that circles S_1 and S_2 interchange and S_3 will be rotated through $180°$.

If we take a disc D which bounds S, then it is easy to see that the corresponding element $w(D) \in W(S) = A$ is a unit of the algebra A.

(3.2.3) We can interpret the element $w(D)$ as a functional $\eta : A \to \mathbb{C}$. Then, clearly, $\theta(a, b) = \eta(ab)$.

(3.2.4) It is easy to check that the algebra A equipped with the form θ (or equivalently with the functional η) allows uniquely reconstruct TFT W.

In examples the algebra A is usually semisimple, i.e. isomorphic to $\bigoplus_{i=1}^{k} \mathbb{C}$.

(3.2.5) **Exercise.** $\dim A = w(T^2)$, where T^2 is the torus $S \times S$.

(3.2.6) Now consider a $(2,1)$-dimensional TFT W. Starting with it we can produce a $(1,1)$-dimensional TFT V by $V(L) = W(S \times L)$, where S is the standard circle. As we saw, such theory V can be described by a commutative algebra $A = V(S) = W(S \times S)$. This algebra A is called the *fusion algebra* of TFT W.

3.3. $(2,1)$-dimensional theories.

(3.3.1) How to give examples of $(2,1)$- dimensional topological field theories, and hence construct invariants of 3-manifolds ? Let us try to analyze the situation.

For a start we are mostly interested in level 0 description, i.e. to every 3-manifold N we would like to assign an invariant $w(N)$.

We can try to do it passing to level 1 theory. Then instead of very complicated objects – 3-manifolds – we have to study much more manageable objects – 2-manifolds. However, we have to pay for this simplification: now we assign to a manifold not a number, but an algebraic object – a vector space.

It turns out, that we can move father in the same direction. Namely, we can pass to 1-manifolds, provided we assign to them even more complicated structures. This naturally leads us to level 2 description of TFT.

(3.3.2) **Definition of TFT on level 2.**

I. A TFT W of dimension $(d, 1)$ assigns to every closed $(d-1)$-manifold L an abelian category $\mathcal{W}(L)$ of finite type over $\mathbb{C}$ (see below).

II. To every bordism M between manifolds L_1 and L_2 TFT W assigns a functor $W(M) : \mathcal{W}(L_1) \to \mathcal{W}(L_2)$.

III. Suppose we are given three $(d-1)$-manifolds L_1, L_2, L_3 and bordisms M' between L_1 and L_2 and M'' between L_2 and L_3. Consider a composit bordism M between L_1 and L_3. Then TFT W should provide an isomorphism $\alpha_{M', M''} : W(M) \xrightarrow{\sim} W(M'') \circ W(M')$.

IV. Let M, M' be two bordisms between L_1 and L_2. Suppose N is a bordism between M and M'. In this situation TFT W should assign to N a morphism of functors $w(N) : W(M) \to W(M')$.

(3.3.3) An abelian category of finite type over $\mathbb{C}$ may be defined as a category equivalent to the category of finite dimensional B-modules for some finite dimensional $\mathbb{C}$-algebra B.

In fact in all known cases of TFT the algebra B is semisimple, so the correspoding category is equivalent to the category of vector bundles on a finite set.

For categories of finite type one naturally defines their tensor product (3.3.4) We assume that the correspondance $L \mapsto \mathcal{W}(L)$ in (3.3.2) I is functorial with respect to diffeomorphisms of L. This means that to every diffeomorphism φ it assigns a functor $W(\varphi)$, and to every composition of diffeomorphisms $\varphi \circ \psi$ it assigns an isomorphism of functors $W(\varphi \circ \psi) \tilde{\rightarrow} W(\varphi) \circ W(\psi)$. These isomorphisms should satisfy some form of pentagon identity.

Similarly in (3.3.2) II we assume that the correspondence $M \mapsto W(M)$ is functorial with respect to diffeomorphisms of M.

We also assume that the correspondance W is multiplicative.

(3.3.5) The data described in 3.3.2 should satisfy many compatibility conditions. For example, there should be a pentagon type identity when we consider a composition of three bordisms M_1, M_2, M_3.

When one attempts to list all these conditions, he ends up with many pages of definitions and verifications of their compatibilities. In fact by working with an appropriate notion of stalk of manifolds one can reduce this description to a manageable size.

We will not try to list all these conditions and to give a rigorous definition of level 2 TFT. Instead let us fix a $(2,1)$-dimensional TFT W and try to describe what structures lie behind this notion.

(3.3.6) Consider the abelian category $C = \mathcal{W}(L)$, where $L = S$ is the standard circle. It turns out that the requirement that $\mathcal{W}(S)$ functorially depends on S and the homotopy invariance property imply that the category C has an additional structure, namely an automorphism of the identity functor $t : \mathrm{Id}_C \to \mathrm{Id}_C$.

Indeed consider the family of rotations $\varphi_u : S \to S$, $u \in [0,1]$, where φ_u is the rotation of S through the angle $2\pi u$. Fix an object $X \in C$. Then for the same reasons as in 3.1.8. the family of objects $X_u = W(\varphi_u)(X) \in C$ should be locally constant. Since the equality of objects does not make sense, this just means that X_u is a local system of objects of category C on the segment $[0,1]$ (e.g. if $C = \mathrm{Vec}$ is the category of vector spaces, this will be a usual local system on the segment). This local system defines canonical isomorphisms between all objects X_u. On the other hand, since the diffeomorphism φ_1 is identity, we have $X_1 = X$. This defines a monodromy operator $t : X = X_0 \tilde{\rightarrow} X_1 = X$.

Later we will use the following equivalent description of the automorphism t. Consider the functor $P = W(\varphi_{1/2}) : C \to C$ and the isomorphism $p : Id \to P$ described above. Then P^2 equals (i.e. is canonically isomorphic) to the identity functor Id, so the morphism p defines an isomorphism $p^2 : Id \to P^2 = Id$, which we denote by t.

(3.3.7) Now consider pants as in (3.2.2), which define the bordism M between $S \cup S$ and S. Then the TFT W assigns to M a functor $F = W(M) : C \times C \to C$. This functor defines on C the structure of a monoidal category.

Namely, we define the associativity constraint as an isomorphism corresponding to the natural diffeomorphism $M \circ (M \circ M) \tilde{\to} (M \circ M) \circ M$. The pentagon identity follows from the fact that two composite diffeomorphisms between products of three bordisms are isotopic.

(3.3.8) Let φ be a rotation of M through 180! as in (3.2.2). Then φ defines an isomorphism $\varphi_t(X \otimes Y) \tilde{\to} Y \otimes X$.

A choice of a half of the third circle S defines a canonical isomorphism $\varphi_t(X \otimes Y) \tilde{\to} X \otimes Y$, and hence an isomorphism $S_{XY} : X : \otimes Y \tilde{\to} Y \otimes X$. It is easy to see that S_{XY} defines a symmetry constraint and that

$$S_{XY} \circ S_{YX} = T_{X \otimes Y} \circ T_X^{-1} \circ T_Y^{-1} .$$

(3.3.8) Now let us define a symmetry constraint S on C.

Let φ be the rotation of pants M through 180° (as in (3.2.2)). Th $W(\varphi)$ defines an isomorphism

$$W(\varphi) : P(X \otimes Y) \tilde{\to} P(Y) \otimes P(X) ,$$

where $P : C \to C$ is the functor, corresponding to the rotation of S through 180° (see (3.3.6)).

Using an isomorphism $p : \mathrm{Id} \to P$, described in (3.3.6) we can interpret $W(\varphi)$ as an isomorphism

$$S_{X,Y} : X \otimes Y \to Y \otimes X .$$

Rotation φ extends to a diffeomorphism $\varphi : N \circ (N \circ N) \tilde{\to} (N \circ N) \circ N$, which implies that the isomorphism $W(\varphi)$ is compatible with the associativity constraint. Rewriting this in term of S_{XY} gives hexagon identities for the symmetry constraint S.

(3.3.9) In order to carry constructions described in (3.3.6) - (3.3.8) we need only part of the data, defining TFT, namely data I, II, III. So, let us define a *modular functor W* (of dimension (2.1)) as a correspondence

I.	$\{1 \text{ manifold } L\}$	$\Longrightarrow$ acategory$\mathrm{W}(L)$.
II.	$\left\{\begin{array}{l}\text{Bordism } M \\ \text{between } L_1, L_2\end{array}\right\}$	$\Longrightarrow$ a functor $W(M) : \mathcal{W}(L_1) \to \mathcal{W}(L_2)$.
III.	$\left\{\begin{array}{l}\text{Bordisms } M', M'' \\ \text{between } L_1 \text{ and } L_2 \text{ and} L_2 \text{ and} L_3\end{array}\right\}$	$\Longrightarrow$ an isomorphism $a_{M',M''} : W(M) \tilde{\to} W(M'') \circ W(M')$.

Then the constructions described in (3.3.6) - (3.3.8) show that modular functor W defines a balanced tensor category on $C = \mathcal{W}(S)$.

(3.3.10) In fact we do not need to know values of the modular functor W on all bordisms.

Let us say that a bordism M has genus 0 if the closed surface M obtained from M by gluing discs to all circles on its boundary is diffeomorphic to a sphere.

Let us define a *genus 0 modular functor* as a correspondence I, II, III whi is defined only when all bordisms are of genus 0. Then as in (3.3.6) - (3.3.8) w check that genus 0 modular functor W defines a balanced tensor category.

It turns out that any balanced tensor category uniquely corresponds to a genus 0 modular functor. In other words an algebraic notion of balanced tensor category the same as a geometric notion of a genus 0 modular functor.

This fact, which is implicitly contained in Drinfeld's paper on quasi-Hopf algebras, was explicitly formulated by Deligne in his letter to Drinfeld (Delign used slightly different but equivalent language of punctured algebraic curves).