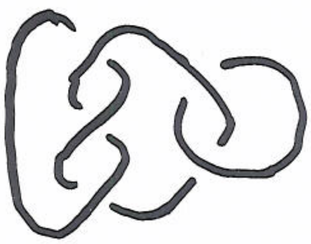
Link homology

M. Khovanov

ICM talk, Madrid, 2006

Link homology

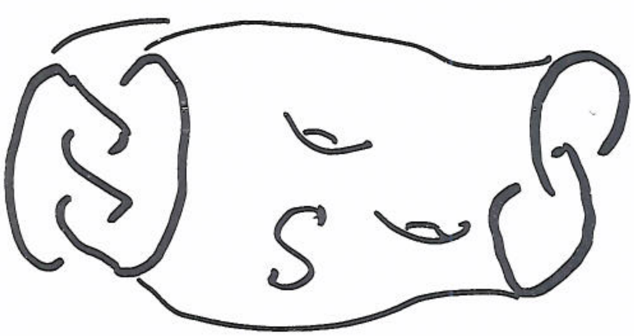
A link $L \subset \mathbb{R}^3$

L 

Homology groups $H(L)$

A link cobordism

$S \subset \mathbb{R}^3 \times [0,1]$



A homomorphism

$H(S): H(L_0) \longrightarrow H(L_1)$

$L_0 = S \cap \mathbb{R}^3 \times \{0\}$

$L_1 = S \cap \mathbb{R}^3 \times \{1\}$

H is a functor

Category of link cobordisms $\xrightarrow{\ H\ }$ Category of (graded) abelian groups / vector spaces / etc

A (3+1)-dimensional TQFT restricted to links and link cobordisms

$h(L) = \chi(H(L))$ Euler characteristi

$h(L)$ a link invariant

H is a categorification of h

HOMFLY-PT Polynomial

$$a\,P\Big($$  $$\Big) - a^{-1}P\Big($$  $$\Big) = (q-q^{-1})P\Big($$ 

$$P(L) \in \mathbb{Z}[a^{\pm 1}, (q-q^{-1})^{\pm 1}]$$

$$a = q^n \qquad P_n(L) \in \mathbb{Z}[q, q^{-1}]$$

$n=0$ Alexander polynomial — knot Floer homology

$$H_0(L) = \bigoplus_{i,j} H_0^{i,j}(L)$$

$$P_0(L) = \sum_{i,j} (-1)^i q^j \,\mathrm{rk}\, H_0^{i,j}(L)$$

P. Ozsváth, Z. Szabó, J. Rasmussen

$n=1$ trivial case $\quad P_1(L) = 1 \quad \forall L$

$n=2$ Jones polynomial $P_2(L) = J(L)$

$$H_2(L) = \bigoplus_{i,j} H_2^{i,j}(L)$$

$$P_2(L) = \sum_{i,j} (-1)^i q^j \operatorname{rk} H_2^{i,j}(L)$$

M.K. math.QA/9908171

$n=3$ $H_3(L)$ M.K. math.QA/0304375

any $n>1$ $H_n(L)$ Lev Rozansky, M.K. math.QA/0401268

$$H_n(L) = \bigoplus_{i,j} H_n^{i,j}(L)$$

$$P_n(L) = \sum_{i,j} (-1)^i q^j \operatorname{rk} H_n^{i,j}(L)$$

triply-graded theory $H(L)$

$P(L) = \chi(H(L))$

Lev Rozansky, M.K math.QA/0505056

Restrict a link homology theory to trivial links and trivial cobordisms

Get a two-dimensional TQFT

$H(\emptyset) = R$ commutative ring

$H(\bigcirc) = A$ commutative Frobenius R-algebra

$$H(\underbrace{\bigcirc\bigcirc\ldots\bigcirc}_{k}) = A^{\otimes k}$$

$$\begin{tikzcd} R \arrow[d, "\iota"] \\ A \end{tikzcd}$$

unit map

$$\begin{tikzcd} A^{\otimes 2} \arrow[d, "m"] \\ A \end{tikzcd}$$

commutative, associative multiplication

$$\begin{tikzcd} A \arrow[d, "\varepsilon"] \\ R \end{tikzcd}$$

trace map

A categorification of the Jones polynomial

$H(\emptyset) = \mathbb{Z}$ $\qquad\qquad$ R

$H(O) = \mathbb{Z}[x]/(x^2)$ $\qquad\qquad$ A

$\varepsilon(x) = 1, \quad \varepsilon(1) = 0$ $\qquad\qquad$ $\varepsilon : A \longrightarrow R$

trace map

make A graded

$\deg(1) = -1 \qquad \deg(x) = 1$

$$\begin{array}{ccc}
\deg & R & A \\
1 & & \mathbb{Z}\cdot x \\
0 & \mathbb{Z}\cdot 1 & \\
-1 & & \mathbb{Z}\cdot 1
\end{array}$$

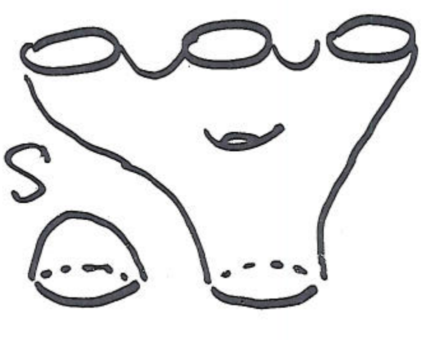
$$\mathbb{Z}\cdot x \xrightarrow{\varepsilon} \mathbb{Z}\cdot 1 \qquad \mathbb{Z}\cdot 1 \xrightarrow{i} \mathbb{Z}\cdot 1$$

$\deg(i) = -1$

$\deg(\varepsilon) = -1$

$\deg(m) = 1$

$\deg H(S) = -\chi(S)$



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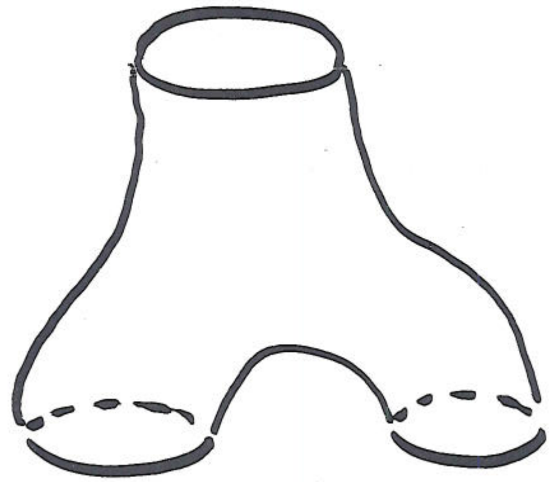


$$A^{\otimes 2} \xrightarrow{m} A$$

$$1 \otimes X \longmapsto X$$

$$X \otimes X \longmapsto 0$$

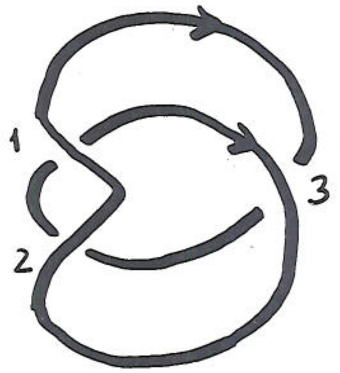
$$A \xrightarrow{\Delta} A^{\otimes 2}$$

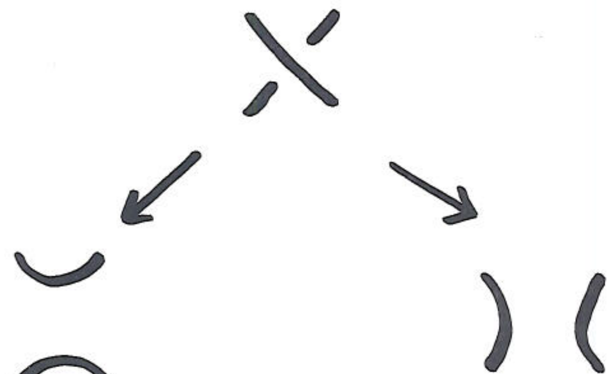
$$1 \overset{\Delta}{\longmapsto} X \otimes 1 + 1 \otimes X$$

$$X \longmapsto X \otimes X$$

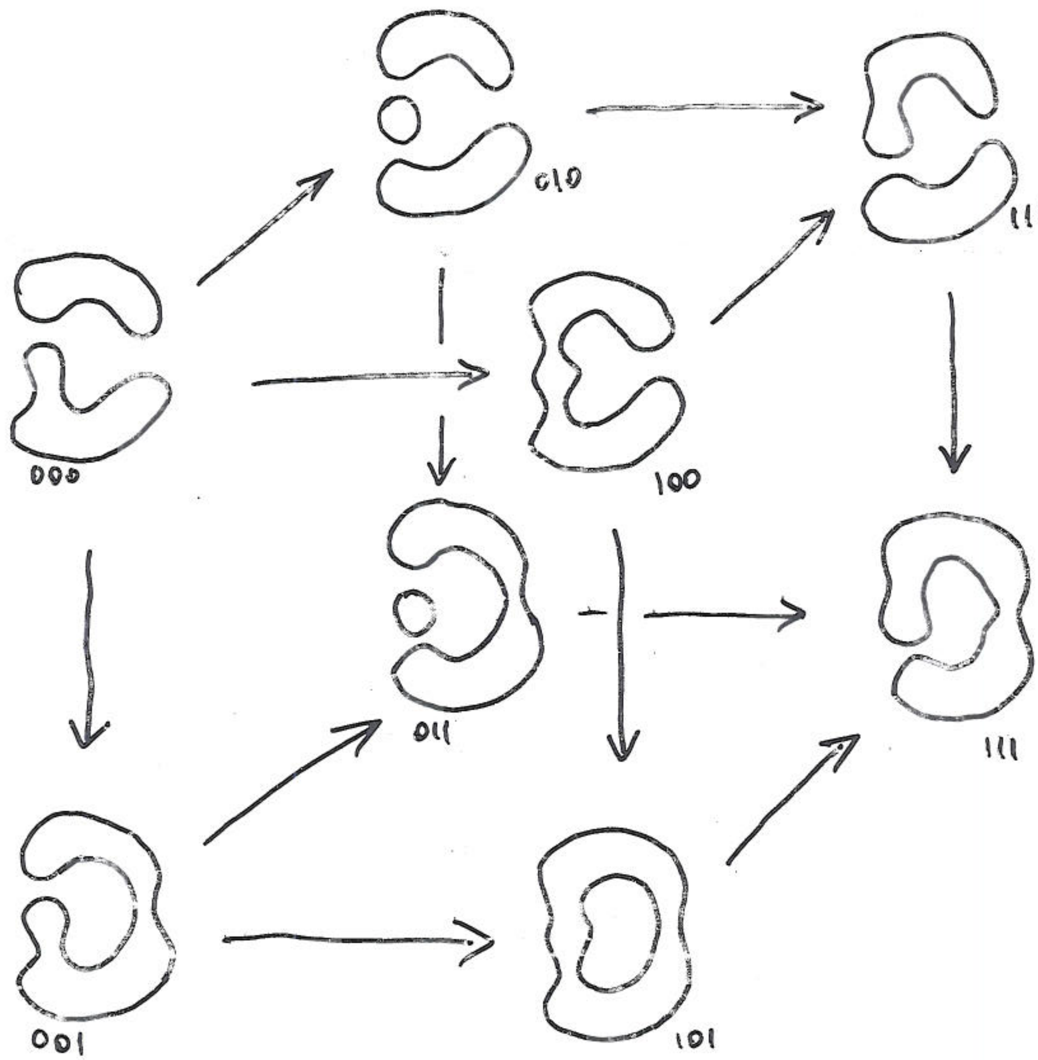
$$\deg m = \deg \Delta = 1$$

D plane diagram of oriented link L



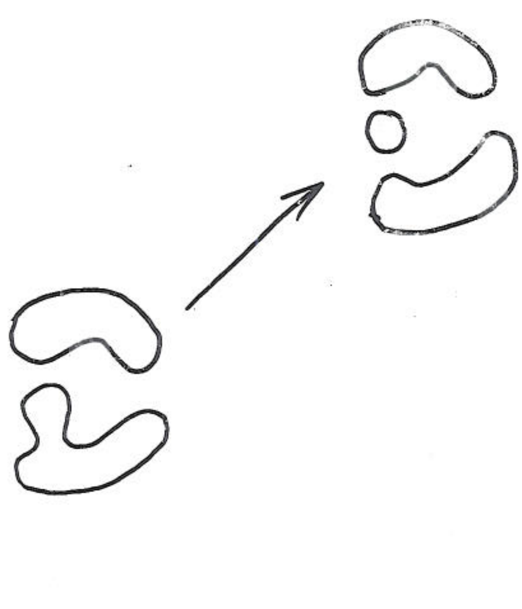


0-smoothing   1-smoothing



Commutative cube

$$\begin{tikzcd}
& A^{\otimes 3} \arrow[rr, "m"] \arrow[dd, "m"] & & A^{\otimes 2} \arrow[dd, "m"] \\
A^{\otimes 2} \arrow[ur, "\Delta"] \arrow[rr, "m"] \arrow[dd, "m"] & & A \arrow[ur, "\Delta"] \arrow[dd, "\Delta"] & \\
& A^{\otimes 2} \arrow[rr, "m"] & & A \\
A \arrow[ur, "\Delta"] \arrow[rr, "\Delta"] & & A^{\otimes 2} \arrow[ur, "m"] &
\end{tikzcd}$$



$$A^{\otimes 2} \xrightarrow{\mathrm{Id} \otimes \Delta} A^{\otimes 3}$$

Total complex

$$0 \to A^{\otimes 2} \xrightarrow{d} \begin{array}{c} A^{\otimes 3}\{-1\} \\ \oplus \\ A\{-1\} \\ \oplus \\ A\{-1\} \end{array} \xrightarrow{d} \begin{array}{c} A^{\otimes 2}\{-2\} \\ \oplus \\ A^{\otimes 2}\{-2\} \\ \oplus \\ A^{\otimes 2}\{-2\} \end{array} \xrightarrow{d} A\{-3\} \xrightarrow{d} 0$$

$-x(D)$ $\qquad\qquad\qquad\qquad$ $y(D)$

$x(D) = \#$ 

$y(D) = \#$ 

add overall shift $\{2x(D) - y(D)\}$

Complex $C(D)$ $\qquad$ $C(D) = \bigoplus_{i,j} C^{i,j}(D)$

$$C^{i,j}(D) \xrightarrow{d} C^{i+1,j}(D)$$

Homology $\quad H(D) = \bigoplus_{i,j} H^{i,j}(D)$

link L $\longrightarrow$ diagram D

Proposition $\chi(H(D)) = \sum_{i,j} (-1)^i q^j \operatorname{rk} H^{i,j}(D)$

is the Jones polynomial of L

Kauffman's approach to Jones polynomial

$\langle D \rangle \in \mathbb{Z}[q, q^{-1}]$

$$\langle \times \rangle = \langle \asymp \rangle - q^{-1} \langle)\,(\rangle$$

$$\langle \underbrace{\bigcirc\!\!\circ \;\; \bigcirc}_{k \text{ circles}} \rangle = (q + q^{-1})^k$$

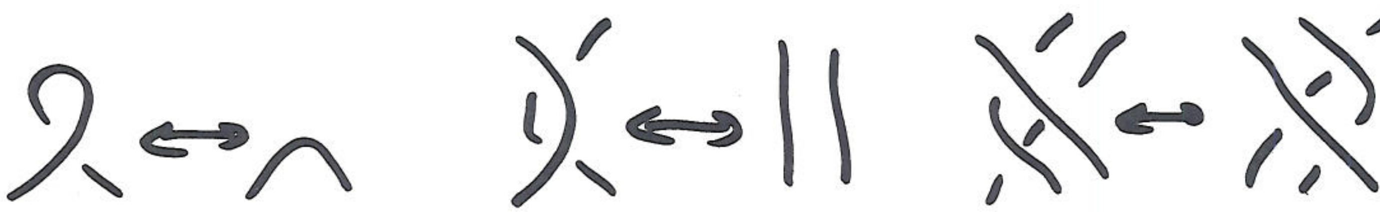
$$J(L) = (-1)^{x(D)} q^{2x(D) - y(D)} \langle D \rangle$$

<u>Theorem</u> (M.K.) $H^{i,j}(D)$ are invariants of L only.

$$H(L) = \bigoplus_{i,j} H^{i,j}(L)$$

if D_1, D_2 are related by a Reidemeister move, construct a homotopy equivalence

$$C(D_1) \simeq C(D_2)$$



Theorem (M. Jacobsson ; M.K)
$H(L)$ is functorial. A link cobordism S between L_0 and L_1 induces a homomorphism $H(S): H(L_0) \to H(L_1)$ well-defined up to overall minus sign.

S. Morrison, K. Walker. How to decorate to get rid of the minus sign.

Applications

1. J. Rasmussen, E. S. Lee

A combinatorial proof of the Kronheimer - Mrowka's theorem (Milnor's conjecture)

$$g_4(T_{n,m}) = \frac{(n-1)(m-1)}{2}$$

if K - positive knot 

then $g_4(K) = g(K)$

slice genus (K) = Seifert genus (K)

$X^2 = 0 \longrightarrow X^2 = t$ t - formal parameter

D. Bar-Natan

$H_t(L) \otimes_{\mathbb{Z}} \mathbb{Q}$ $\mathbb{Q}[t]$-module

modulo t-torsion, $\mathbb{Q}[X]\{s(L)-1\}$

$s(L)$ knot concordance invariant

$$g_4(L) \geq \frac{|s(L)|}{2}$$

2. Lenny Ng, Hao Wu

$H(L)$ gives an upper bound for the Thurston-Bennequin number of a link.

sharp for all but at most two knots with ≤ 10 crossings.

3. P. Ozsváth, Z. Szábó

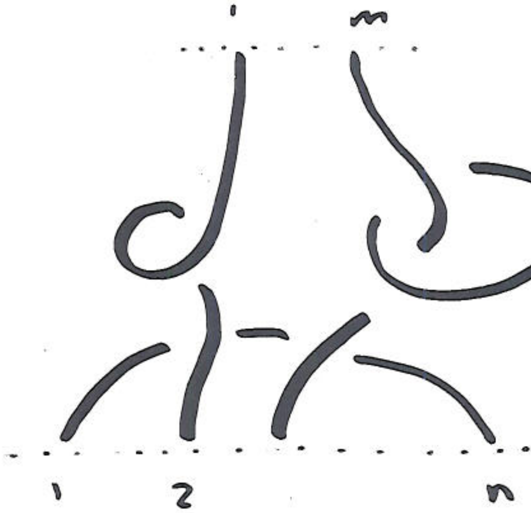
$$\overline{H}(K, \mathbb{Z}_2) \Longrightarrow \widehat{HF}\left(\begin{array}{c}\text{branch}\\ \text{double cover}\\ \text{of } K\end{array}\right)$$

$$E^2 \qquad\qquad E^\infty$$

K H-thin $\Rightarrow$ $D(K)$ is L-space

Problem: find more applications.

links $\longrightarrow$ tangles



(m,n)-tangle

Category of tangles

objects: $0, 1, 2, 3 \ldots$

morphisms $n \to m$

Jones polynomial extends to an invariant of tangles

V 2-dimensional defining representation of $U_q(sl_2)$

$$\text{tangles} \xrightarrow{\ J\ } U_q(sl_2)\text{-representations}$$

$$n \longmapsto V^{\otimes n}$$

$$T \longmapsto J(T) : V^{\otimes n} \to V^{\otimes m}$$

A more economical approach:

restrict to even m, n

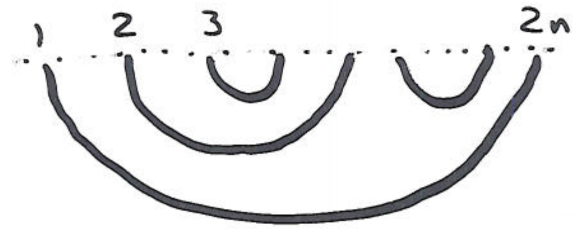
$(2m, 2n)$-tangles

$$2n \longmapsto \mathrm{Inv}(V^{\otimes 2n}) = \mathrm{Hom}_{U_q(sl_2)}(\mathbb{C}, V^{\otimes 2n})$$

$$T \longmapsto J_{inv}(T)$$

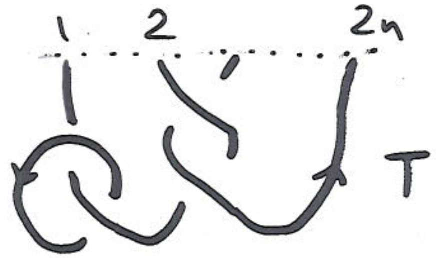
$$\begin{array}{ccc} V^{\otimes 2n} & \xrightarrow{J(T)} & V^{\otimes 2m} \\ \cup & & \cup \\ \mathrm{Inv}(V^{\otimes 2n}) & \xrightarrow{J_{inv}(T)} & \mathrm{Inv}(V^{\otimes 2m}) \end{array}$$

Basis in $\mathrm{Inv}(V^{\otimes 2n})$



crossingless matchings of $2n$ points

$$\dim \mathrm{Inv}(V^{\otimes 2n}) = \frac{1}{n+1}\binom{2n}{n} = \text{n-th Catalan number}$$



$J_{inv}(T) \in Inv(V^{\otimes 2n})$

How to categorify $J_{inv}(T)$?

numbers $\longrightarrow$ vector spaces / homology groups

vector spaces $\longrightarrow$ categories

$G_0(A) \longleftarrow\!\shortmid$ A - abelian category

Grothendieck group of A

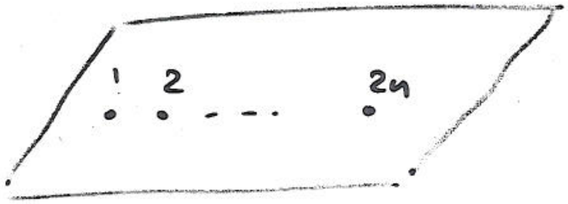
generators $[M]$, $M \in Ob(A)$

relations $[M_2] = [M_1] + [M_3]$

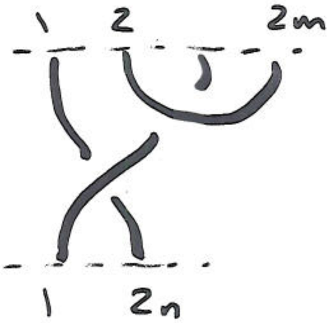
$$0 \to M_1 \to M_2 \to M_3 \to 0$$

$C(A)$ a triangulated category built out of A

Category of bounded complexes of objects of A modulo chain homotopies.



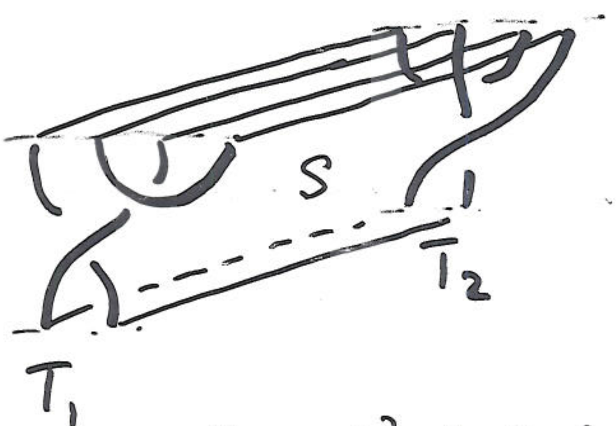
$\longmapsto$ A triangulated category C_n



T $\longmapsto$ An exact functor

$$C_n \xrightarrow{F(T)} C_n$$

$T \subset \mathbb{R}^2 \times [0,1]$



$\longmapsto$ A natural transformation of functors

$$F(T_1) \xRightarrow{F(S)} F(T_2)$$

$S \subset \mathbb{R}^2 \times [0,1] \times (0,1)$

S - tangle cobordism

F is a 2-functor

2-category of tangle cobordisms $\longrightarrow$ 2-category of natural transformations between exact functors

$C_n = C(A_n\text{-gmod})$

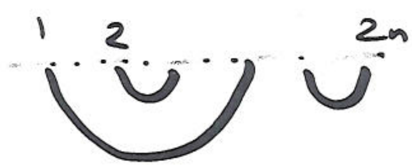
complexes of graded modules over certain ring A_n

a

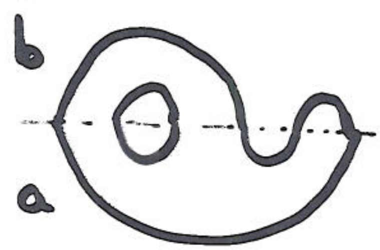
$\longmapsto$ P_a indecomposable projective A_n-module

crossingless matching

$$A_n = \bigoplus_{a,b} \mathrm{Hom}(P_a, P_b)$$



b



k circles

$$\mathrm{Hom}(P_a, P_b) \simeq A^{\otimes k}$$

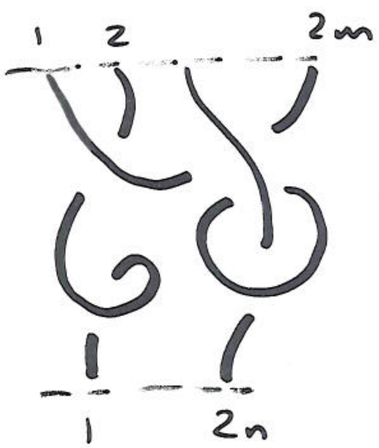
before : $a \longmapsto$ a vector in $\mathrm{Inv}(V^{\otimes 2n})$

now : $a \longmapsto$ an object of A_n-gmod

T

$$0 \longrightarrow P_a \longrightarrow P_b\{-1\} \longrightarrow 0$$

$F(T)$ is a complex of graded projective A_n-modules $F(T) \in Ob(C_n)$



T

$F(T)$ is a complex of graded (A_m, A_n)-bimodules

tensoring with $F(T)$ is a functor

$$C_n \longrightarrow C_m$$

$$\text{Grothendieck group}(A_n) \otimes \mathbb{C} \simeq \mathrm{Inv}(V^{\otimes 2n})$$

$$C_n \xrightarrow{F(T)} C_m \qquad \text{functor}$$

$$[F(T)] = J_{inv}(T)$$

$$\mathrm{Inv}(V^{\otimes 2n}) \longrightarrow \mathrm{Inv}(V^{\otimes 2m})$$

$$\mathrm{Center}(A_n) \simeq H^*(\text{Springer fiber})$$

All flags in $\mathbb{C}^{2n}$

$$0 \subset L_1 \subset L_2 \ldots \subset L_{2n-1} \subset \mathbb{C}^{2n}$$

$$\mathcal{N} L_i \subset L_{i-1}$$

$$\mathcal{N} = \begin{array}{cc} \left(\begin{array}{c|c} \begin{matrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{matrix} & \\ \hline & \begin{matrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{matrix} \end{array}\right) & \begin{matrix} n \\ \\ \\ \\ n \end{matrix} \\ n \qquad\qquad n & \end{array}$$

Springer fiber $\subset$ Quiver variety $\mathcal{M}_n$

$n=1$ $\qquad \mathbb{P}^1 \subset T^*\mathbb{P}^1$

$n>1$ $\qquad$ singular $\subset$ smooth

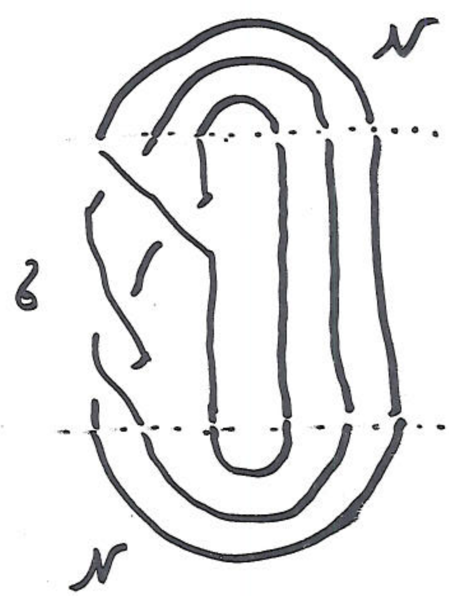
P. Seidel, I. Smith

Braid group acts on Fukaya-Floer category of Lagrangians in $\mathcal{M}_n$.

Link homology theory

$$\widetilde{H}(L) = HF(\mathcal{N}, \beta\mathcal{N})$$

$L = \hat{\beta}$ closure of braid β



Conjecture: $\widetilde{H}(L) = H(L)$

$$C_n \subset FF(\mathcal{M}_n)$$

Categorification of $V^{\otimes n}$ $\qquad$ $U_q(sl_2)$

$$V^{\otimes n} = \bigoplus_{k=0}^{n} V^{\otimes n}(k)$$ 2k-n weight space

$V^{\otimes n}(k)$ = Groth. group $(\mathcal{O}^{k,n-k})$

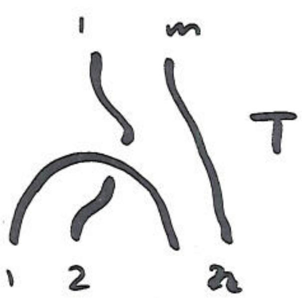
$\mathcal{O}^{k,n-k}$ a parabolic block of highest weight category for $sl(n)$

∞-dim reps of $sl(n)$

simples $(\mathcal{O}^{k,n-k}) = \binom{n}{k}$

$$\mathcal{O}^{n} = \bigoplus_{k=0}^{n} \mathcal{O}^{k,n-k} \qquad G_0(\mathcal{O}^n) \otimes \mathbb{C} \simeq V^{\otimes n}$$

J. Bernstein, I. Frenkel, M.K



$$F(T) : D^b(\mathcal{O}^n) \longrightarrow D^b(\mathcal{O}^n)$$

C. Stroppel

an invariant of tangle cobordisms

$\mathrm{Inv}(V^{\otimes 2n}) \subset V^{\otimes 2n}$ $\qquad$ $C_n \longrightarrow D^b(\mathcal{O}^n)$

highest weight categories,
affine & cyclotomic Hecke
algebras,
geometric representation theory,
Langlands program,
Categorification of
Lie algebra and
quantum group
representations

Fukaya - Floer
categories of quiver
varieties; coherent
sheaves on these
varieties

Link
homology

Matrix factorizations,
LG-models of
string theory

Knot Floer homology
OS & SW theories
4D TQFTs

Euler
characteristic

Quantum invariants of links
Jones, Alexander, HOMFLY-PT polynomials
Chern-Simons theory
Topological strings