A functor \( F: C \to C \) is exact if it takes short exact sequences to short exact sequences:

\[
0 \to M_1 \to M_2 \to M_3 \to 0
\]

\[
\downarrow
\]

\[
0 \to FM_1 \to FM_2 \to FM_3 \to 0
\]

\( F \) induces an endomorphism \([F] \) of \( \text{End}_C(C) \) taking \([M] \) to \([FM] \)
A categorification of the ring of polynomials \( \mathbb{Q}[x] \).

Differentiation \( \partial \) and multiplication by \( x \) are operators on \( \mathbb{Q}[x] \), and

\[
\partial x = x \partial + 1
\]

Have two integral lattices

\( \mathbb{Z}[x] \) and \( \mathbb{Z} \left[ \frac{x^n}{n!} \right] \)

stable under \( x \) and \( \partial \).
Have a bilinear form $\langle , \rangle$

$\mathbb{Z}[x] \times \mathbb{Z}[\frac{x^n}{n!}] \rightarrow \mathbb{Z}$

$\langle x^m, x^n \rangle = \delta_{n,m} \frac{n!}{n!}$

$\langle x \cdot f, g \rangle = \langle f, \partial g \rangle$

After categorification, $\langle , \rangle$ will become $\text{Hom}(\cdot, \cdot)$ and $x, \partial$ will become adjoint functors

$\text{Hom}(X M, N) \cong \text{Hom}(M, DN)$
$T_i - Newton divided difference operator on polynomials$

$f \in C[y_1, y_2, \ldots, y_n]$

$Si f - permute y_i and y_{i+1}$ in $f$

$S_i (y_1^3, y_2^3) = y_2^3 y_1$

$T_i f = \frac{f - Si f}{y_i - y_{i+1}}$

$T_i (y_1^3, y_2^3) = \frac{y_1^3 y_2^3 - y_1 y_2^3}{y_i - y_{i+1}} = \frac{y_1^3 y_2 - y_1 y_2^3}{y_1 - y_2} = y_1 y_2 (y_1^2 + y_2^2)$
$R_n$ - nil-Coxeter algebra/field $k$

generators $T_1, T_2, \ldots, T_{n-1}$

relations

$T_i^2 = 0$

$T_i T_j = T_j T_i \quad |i-j|>1$

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

$\dim R_n = n!$

Has unique simple module

$L_n = k v$ , $T_i v = 0$

$G_0(R_n) \approx \mathbb{Z}$, generators $[L_n]$
$G_0(R_n{-}\text{mod}) \cong \mathbb{Z} \frac{x^n}{n!}$

$[\mathcal{A}_n] \mapsto \frac{x^n}{n!}$

$R_n$ has composition series with $n!$ copies of $\mathcal{A}_n$

$[R_n] = n! \ [\mathcal{A}_n]$

$[R_n] \mapsto x^n$

Have inclusions $R_n \subseteq R_{n+1}$

Induction and restriction functors

$R_n{-}\text{mod} \xrightarrow{\text{Ind}} R_{n+1}{-}\text{mod}$

$\xleftarrow{\text{Res}}$
A ≤ B inclusion of rings

\[ \text{Ind} : A\text{-mod} \rightarrow B\text{-mod} \]

\[ \text{Ind} (M) = B \otimes_A M \]

\[ \text{Res} : B\text{-mod} \rightarrow A\text{-mod} \]

\[ \text{Res} (N) = A^N \]

Restriction is an exact functor.

Induction is exact sometimes.

If B is projective as right A-module, Ind is exact.
\[ C = \bigoplus_{n \geq 0} R_n \text{-mod} \]

\[ G_0(C) = \bigoplus_{n \geq 0} G_0(R_n \text{-mod}) \]

\[ G_0(C) \cong \mathbb{Z} \left[ \frac{x^n}{n!} \right]_{n \geq 0} \]

\[ X(R_n) = R_{n+1} \]

\[ [R_n] \xrightarrow{[x]} [R_{n+1}] \]

\[ x^n \quad \xrightarrow{\text{induction}} \quad x^{n+1} \]

\[ [X] \text{ is multiplication by } x \]
\[ D(R_{n+1}) = R_n \oplus n+1 \]

\[ x^{n+1} \longrightarrow (n+1) x^n \]

\[ [R_{n+1}] \longrightarrow [R_n \oplus n+1] \]

[D] is differentiation \( D \)

\[ D(\alpha_{n+1}) = \alpha_n \]

\[ \frac{x^{n+1}}{(n+1)!} \longrightarrow \frac{x^n}{n!} \]

\[ \text{Hom}_c(R_n, \alpha_m) = \begin{cases} 0, & n \neq m \\ \mathbb{C}, & n = m \end{cases} \]

\[ (x^n, \frac{x^m}{m!}) = \delta_{n,m} \]

\[ \dim \text{Hom}_c(P, M) = \langle [P], [M] \rangle \]

\( P \) - projective
\[ C \xrightarrow{X, D} C \quad C = \bigoplus_{n \geq 0} R_n\text{-mod} \]

\[ G_0(C) \xrightarrow{[X], [D]} G_0(C) \]

\[ \mathbb{Z} \left[ \frac{x^n}{n!} \right] \xrightarrow{x, \partial} \mathbb{Z} \left[ \frac{x^n}{n!} \right] \]

**Proposition** There is an isomorphism of functors

\[ DX = XD \oplus \text{Id} \]

\[ R_n\text{-mod} \rightarrow R_n\text{-mod} \]

\[ \partial x = x\partial + 1 \]
\[ R_n \otimes R_m \subset R_{n+m} \]

Ind, Res functors

Sum over all \( n, m \geq 0 \)

\[ C \otimes C \xrightarrow{\text{Ind}} C \xleftarrow{\text{Res}} C \]

\[ G_0(C) \times G_0(C) \xrightarrow{\text{[Ind]}} G_0(C) \xleftarrow{\text{[Res]}} G_0(C) \]

\[ \mathbb{Z} \left[ \frac{x^n}{n!} \right] \times \mathbb{Z} \left[ \frac{x^n}{n!} \right] \xrightarrow{\text{multiplication}} \mathbb{Z} \left[ \frac{x^n}{n!} \right] \]

\[ x^n \otimes x^m \xrightarrow{\text{comultiplication}} x^{n+m} \]

\[ \sum_{k=0}^{n} \frac{x^k}{k!} \otimes \frac{x^{n-k}}{(n-k)!} \xrightarrow{\text{comultiplication}} \frac{x^n}{n!} \]