Theorem (M. Jacobsson, M.K.)

Homology theory $H$ extends to a (projective) functor from the category of link cobordisms to the category of bigraded abelian groups.

\[ H(L_0) \rightarrow H(S) \rightarrow H(L_1) \]
\( \mathcal{L}_0, \mathcal{L}_1 \subset \mathbb{R}^3 \)

\( S \subset \mathbb{R}^3 \times [0, 1] \)

\( \exists S = \mathcal{L}_0 \parallel (-\mathcal{L}_1) \)

\( H(s) \) is well-defined up to overall minus sign.

Sign indeterminacy was taken care of by

David Clark
Scott Morrison
Kevin Walker
Applications

Jacob Rasmussen

Combinatorial proof of Kronheimer-Mrowka theorem (Milnor conjecture) on slice genus of positive knots.

\[ M \subset D^4 \quad \subset S^3 \]
\[ \partial M = \Lambda \]

Slice genus \( g_s(\Lambda) \)
Positive

\[
g_s(\mathcal{L}) = \frac{C + 1 - s}{2}
\]

\[\begin{array}{ll}
C = 3 \\
S = 2 \\
\end{array}\]

Lenhazd Ng

Effective upper bound on Thurston - Bennequin number of Legendrian links
HOMFLYPT Polynomial

Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki, Traczyk

\[ P(\lambda) \in \mathbb{Z}[a^{\pm 1}, b^{\pm 1}] \]

\[ a \ P(\uparrow \rightarrow) - a^{-1} \ P(\rightarrow \uparrow) = b \ P(\uparrow \uparrow) \]

\[ a = q^n, \ b = q - q^{-1} \]

\( P(\lambda) \) related to representation theory of \( \mathcal{U}_q(sl(n)) \)

Quantum deformation of \( \mathcal{U}(sl(n)) \)
\[ q^n P_n(\uparrow \rightarrow) - q^{-n} P_n(\leftarrow \rightarrow) = (q-q^{-1}) P_n(\uparrow \downarrow) \]

\( n = 0 \)  Alexander Polynomial

P. Ozsváth, Z. Szabó, J. Rasmussen

Categorification - knot Floer homology

\[
H_0(L) = \oplus_{i,j \in \mathbb{Z}} H_0^{i,j}(L)
\]

\[
P_0(L) = \sum (-1)^i q^d z_k H_0^{i,j}(L)
\]

\( n = 2 \)  Jones polynomial

\[
H_2(L) = \oplus_{i,j \in \mathbb{Z}} H_2^{i,j}(L)
\]

\[
P_2(L) = \sum (-1)^i q^d z_k H_2^{i,j}(L)
\]
any \( n > 2 \)

\[
H_n(L) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}_n(L)
\]

\[
P_n(L) = \sum (-1)^i q^j 2k \ H^{i,j}_n(L)
\]

\[
H_n(O) = \mathbb{Z}[x]/(x^n) \cong H^*(\mathbb{C}P^{n-1}, \mathbb{Z})
\]

\[
P_n(O) = [n] \quad \text{quantum } n
\]

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \ldots + q^{1-n}
\]

Functionality for cobordisms
Alternative approaches to $H_n$ and related theories

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Categories $\rightarrow$ functors $\rightarrow$ Natural transformations

Grothendieck group

Vector spaces $\rightarrow$ free abelian groups $\rightarrow$ operators

Dimension

Numbers
Grothendieck group

$C$ - abelian category

$C = R$-mod , $R$ a ring

$G_0(C)$ - abelian group with generators $[M]$ , $M \in \text{Ob } C$,
relations

$[M_2] = [M_1] + [M_3]$

for each exact sequence

$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$
If $C$ - category of finite length modules over a ring $R$, then $G_0(C)$ is free abelian with basis $[\mathbb{C}]$.

$R = \mathbb{Z}$ basis $[\mathbb{Z}/p]$, $p$ prime

$R = \mathbb{Q}[x]/(x^n)$

$G_0(R\text{-mod}) = \mathbb{Z}$

local ring, has unique simple module $\mathbb{Q}$, $x$ acts by $0$