Graphical calculus, canonical bases and Kazhdan-Lusztig theory

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ABSTRACT THESIS

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The graphical calculus of Penrose-Kauffman describes representation theory of the quantum group $U_q(\mathfrak{sl}_2)$ via plane diagrams. We show that in this interpretation bases dual to Lusztig's canonical bases in tensor products of finite dimensional $U_q(\mathfrak{sl}_2)$-modules have a simple realization and we compute these bases explicitly. We further develop graphical calculus and, in particular, obtain factorization of Clebsch-Gordan and Racah-Wigner coefficients for $U_q(\mathfrak{sl}_2)$ via the intermediate dual canonical basis.

Next, we give formulas for canonical bases via compositions of Jones-Wenzl projectors. We also prove that Lusztig's canonical bases in tensor powers of the two-dimensional fundamental $U_q(\mathfrak{sl}_2)$ representation coincide with the Kazhdan-Lusztig basis for the maximal parabolic (=grassmannian) case of the Weyl group $\mathbb{S}_n$ and its subgroup $\mathbb{S}_k \times \mathbb{S}_{n-k}$. As a byproduct we get a very simple description of Kazhdan-Lusztig basis vectors in the grassmannian case in terms of compositions of Jones-Wenzl projectors.

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INTRODUCTION

In this dissertation we show how the combinatorics of Penrose-Kauffman graphical calculus relates to two fundamental concepts of representation theory: Lusztig's canonical bases and Kazhdan-Lusztig theory. The graphical calculus was invented by Penrose for a diagrammatical description of the tensor category of representations of the simple Lie algebra $\mathfrak{sl}_2$. This was generalized later by Kauffman to the calculus of q-spin networks on the plane, and used by Kauffman and Lins to construct invariants of 3-manifolds. It was soon shown by Piunikhin [Pi] that their invariants coincide with Turaev-Viro invariants [TV]. Reshetikhin-Turaev invariants [RT] are a refinement of Turaev-Viro invariants. With Penrose-Kauffman calculus in hand, Lickorish [Lc] succeeded in providing an elementary construction of Reshetikhin-Turaev invariants of 3-manifolds. The original Reshetikhin-Turaev approach was based on a difficult concept of modular Hopf algebra and required understanding of the representation theory of the quantum group $U_q(\mathfrak{sl}_2)$ at roots of unity.

Piunikhin [Pi] also showed that the q-spin network calculus provides a graphical interpretation of the representation theory of $U_q(\mathfrak{sl}_2)$. In particular, $6j$-symbols and other structure constants for the braided tensor category of $U_q(\mathfrak{sl}_2)$-representations can be derived using q-spin networks (see [CFS], [KaL], [MV]).

Interestingly, the graphical calculus provides natural bases for $U_q(\mathfrak{sl}_2)$-invariants of tensor products of finite-dimensional $U_q(\mathfrak{sl}_2)$-modules. It was conjectured by Greg Kuperberg that these bases are closely related to Lusztig's canonical bases in invariants of tensor products. The precise relation between the two was found by Igor Frenkel and the author [FK]. Namely, slightly generalizing graphical calculus, one can define a graphical basis not only in the invariants, but in the whole tensor product of $U_q(\mathfrak{sl}_2)$-modules. Then it turns out that the graphical basis in a tensor product is dual to Lusztig's canonical basis in the tensor product.

Lusztig's canonical bases are bases in tensor products of finite-dimensional modules over quantum deformations of finite dimensional simple Lie algebras. Originally, canonical bases were discovered by M.Kashiwara [Ks] and G.Lusztig [L1] in nilpotent subalgebras $U_q^-(\mathfrak{g})$ of quantum groups $\mathfrak{g}$ and in finite-dimensional irreducible $U_q(\mathfrak{g})$-modules. Later Lusztig extended this notion to give bases in tensor products of finite-dimensional $U_q(\mathfrak{g})$-modules. He also found other generalizations which we will not be discussing here and instead refer the reader to [L3]. Canonical bases have remarkable properties of positivity and integrality, namely, various structure coefficients of the quantum group (multiplication, the action of generators E_i and F_i) are positive integral. This is explained by the fact that canonical bases come from certain complexes of sheaves on quiver varieties, and, from this point of view, structure coefficients are dimensions of intersection cohomology groups.

Exact formulas for canonical bases are unknown except for a few cases. Most of the work on the canonical basis has been done towards understanding a canonical basis in the nilpotent subalgebra $U_q^-(\mathfrak{g})$ of the quantum group $\mathfrak{g}$ (for example [BFZ],[Ks],[L3]). In this dissertation we go in a different direction: we restrict to the case of $\mathfrak{sl}_2$, but we obtain exact formulas for the canonical and dual canonical bases in arbitrary tensor products of finite-dimensional $U_q(\mathfrak{sl}_2)$-modules. As we already mentioned, the dual canonical basis naturally pops up from the graphical calculus of q-spin networks. Aided by the graphical calculus, we are also able to give formulas for the canonical basis itself. Namely, we show that canonical basis vectors in tensor products are given by compositions of Jones-Wenzl projectors. These are projections from a tensor power of the fundamental representation onto the highest weight irreducible subrepresentation. The intuitive reason for this answer is clear: the dual canonical basis vectors admit a graphical presentation by systems of disjoint arcs, while the Jones-Wenzl projector, composed with an arc, becomes the trivial operator. So, one expects that Jones-Wenzl projectors can be used as building blocks for vectors of the canonical basis– the basis dual to the basis of systems of disjoint arcs.

This intuitive guess is correct and we get simple inductive formulas for canonical basis vectors in tensor products. One can now take these formulas and rewrite them coordinatewise, i.e., write inductive formulas for coefficients of canonical vectors in the standard basis of the tensor product. Surprisingly, in the special case when the tensor product is a tensor power of the fundamental representation, we obtain Zelevinsky's recursive formula for Kazhdan-Lusztig polynomials in the grassmanian case [Z]. That implies the coincidence of Lusztig's canonical basis in tensor powers of the fundamental representation and Kazhdan-Lusztig bases in the grassmanian case.

We prove here that these two bases coincide, using a more invariant approach: by identifying the two vector spaces, the elementary bases of these spaces and showing that under this identification the two involutions – needed for defining canonical, respectively, Kazhdan-Lusztig bases, coincide. The proof is concluded by observing that the integrality properties of the two bases also match. This is a joint result with Igor Frenkel and Alexander Kirillov Jr. [FKK].

This proof does not use the specifics of $\mathfrak{sl}_2$ and can be generalized to establish the coincidence of the canonical basis of a tensor power of the fundamental representation of $\mathfrak{sl}_k$ with the Kazhdan-Lustig basis for the relative case of the symmetric group and its parabolic subgroup with k blocks. We chose not do this general case in our dissertation and the proof will appear in [FKK].

Let us say a few words about the organization of this dissertation. Chapters 1 and 4 contain results of a joint work with Igor Frenkel. Chapter 5 is a joint work with Igor Frenkel and Alexander Kirillov Jr. Chapter 3 contains results of my preprint [Kh]. In Chapter 1 we recall the finite-dimensional representation theory of the quantum group $U_q(\mathfrak{sl}_2)$ and, specializing results of Lusztig [L3] to the $\mathfrak{sl}_2$ case, we define the canonical basis of a tensor product of finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules. Using the invariant form, we define the dual canonical basis in the dual tensor product. Our main results about the exact form of the canonical and dual canonical bases are stated in section 1.4.

In Chapter 2 we recall the graphical calculus of q-spin networks (more details can be found in [CFS], [KaL], [MV]) and, following [FK], construct the dual canonical basis of tensor products. In Chapter 3 we explicitly construct the canonical basis of a tensor product and prove that it is indeed canonical by computing scalar products with the elements of the dual canonical basis.

The standard approach to the graphical calculus treats the Jones-Wenzl projector as an elementary unit and studies networks composed of these projectors. In Chapter 4 we go the other way and look at how the projector itself decomposes in the graphical basis of the Temperley-Lieb algebra. We refine the Jones recursive formula for the projector and in sections 4.1 and 4.2 obtain interesting formulas for the coefficients of the projector in the graphical basis. In the later sections we reap the fruits of our work by giving a very simple non-recursive derivation of formulas for the $6j$-symbol (compare with [MV]) and Clebsch-Gordan coefficients. We intend to write a separate paper where we will apply the results of section 4.2 to Kazhdan-Lusztig theory.

In the last chapter we prove the coincidence of the canonical basis and Kazhdan-

Lusztig basis.

Before we conclude our introduction, let us quickly review the related works by a number of authors. First, books by Carter, Flath, Saito [CFS] and Kauffman, Lins [KaL] are excellent sources to learn about graphical calculus and its relation to representation theory of $U_q(\mathfrak{sl}_2)$. These two books have different emphasis: [KaL] has primary interest in constructing 3-manifold invariants from q-spin networks while [CFS] is mostly about applications of graphical calculus to the representation theory of $U_q(\mathfrak{sl}_2)$.

The geometry of the graphical calculus for $U_q(\mathfrak{sl}_2)$ becomes transparent in the identification of tensor products with homology groups of certain configuration spaces, studied by R.Lawrence and, independently, by A.Varchenko. Under this identification the diagrams representing the dual canonical basis acquire the meaning of actual cycles in these homology groups [FKV].

An attempt to understand Lusztig's canonical bases for tensor products beyond the case of $\mathfrak{sl}_2$ was made in a joint paper with Greg Kuperberg [KhKu]. We compared Kuperberg's graphical bases [Ku] in invariants of tensor products of $\mathfrak{sl}_3$ modules with the bases dual to Lusztig canonical bases. Our result was asymptotically negative: the two bases coincide for tensor products of less then 12 three-dimensional modules and there is exactly one counterexample when there are 12 modules. As the number of factors in the tensor product grows, almost all vectors in the two bases become different.

Fan and Green showed in the preprint [FG] that the image of a certain part of the Kazhdan-Lusztig basis in the Hecke algebra of the symmetric group under homomorphism onto the Templerley-Lieb algebra coincides with the graphical basis of this algebra. It is an interesting question, answer to which is unknown to me at the moment, how to relate their results with ours.

I'd like to conclude this introduction by offering the following problem that came up in a discussion with Arun Ram. The Kazhdan-Lusztig bases in the relative case of the symmetric group $\mathbb{S}_n$ and its maximal parabolic subgroup can be constructed via the graphical calculus of plane diagrams. It is natural to expect that the Kazhdan-Lusztig basis for the relative case of the affine Weyl group $\hat{\mathbb{S}}_n$ and its parabolic subgroup $\mathbb{S}_k \times \mathbb{S}_{n-k}$ can be obtained from the appropriate graphical calculus of diagrams on a cylinder.

CHAPTER I

THE QUANTUM GROUP $U_q(\mathfrak{sl}_2)$ AND ITS REPRESENTATIONS

1.1. The quantum group $U_q(\mathfrak{sl}_2)$

1.1.1 Hopf algebra structure.

Let $\mathbb{C}(q)$ be the field of complex-valued rational functions in an indeterminate q. We denote by $^{-} : \mathbb{C}(q) \to \mathbb{C}(q)$ the $\mathbb{C}$-algebra involution such that $\overline{q^n} = q^{-n}$ for all n.

DEFINITION 1.1. *The quantum group $U_q(\mathfrak{sl}_2)$ is an associative algebra over $\mathbb{C}(q)$ with generators E, F, K, K^{-1} and relations*

$$
\begin{aligned}
&KK^{-1} = 1 = K^{-1}K \\
&KE = q^2 EK \\
&KF = q^{-2} FK \\
&EF - FE = \frac{K - K^{-1}}{q - q^{-1}}
\end{aligned}
\tag{1.1}
$$

The quantum group $U_q(\mathfrak{sl}_2)$ has a Hopf algebra structure with comultiplication Δ

$$
\begin{aligned}
&\Delta K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1} \\
&\Delta E = E \otimes 1 + K \otimes E \\
&\Delta F = F \otimes K^{-1} + 1 \otimes F
\end{aligned}
\tag{1.2}
$$

and counit

$$\eta(K^{\pm 1}) = 1, \quad \eta(E) = \eta(F) = 0$$

There is an explicit expression for the antipode, but we do not need it in this paper. We will often use the shorthand notation $\mathbf{U}$ for $U_q(\mathfrak{sl}_2)$.

Next we will define two types of involutions. We call the first one *the Cartan involution* and denote it by ω:

$$\begin{aligned} &\omega(E) = F, \quad \omega(F) = E, \quad \omega(K^{\pm 1}) = K^{\pm 1}, \quad \omega(q^{\pm 1}) = q^{\pm 1} \\ &\omega(xy) = \omega(y)\omega(x), \quad x, y \in \mathbf{U} \end{aligned} \tag{1.3}$$

The second involution, denoted by σ, will be called *the "bar" involution:*

$$\begin{aligned} &\sigma(E) = E, \quad \sigma(F) = F, \quad \sigma(K^{\pm 1}) = K^{\mp 1}, \quad \sigma(q^{\pm 1}) = q^{\mp 1} \\ &\sigma(xy) = \sigma(x)\sigma(y), \quad x, y \in \mathbf{U} \end{aligned} \tag{1.4}$$

It is easy to check that ω (respectively, σ) is a well-defined antiinvolution (respectively, involution) of $\mathbf{U}$ considered as an algebra over $\mathbb{C}(q)$ (respectively, over $\mathbb{C}$).

Using the bar involution, we define another comultiplication as follows

$$\overline{\Delta}(x) = (\sigma \otimes \sigma)\Delta(\sigma(x)), \quad x \in \mathbf{U}$$

This implies

$$\begin{aligned} &\overline{\Delta} K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1} \\ &\overline{\Delta} E = E \otimes 1 + K^{-1} \otimes E \\ &\overline{\Delta} F = F \otimes K + 1 \otimes F \end{aligned} \tag{1.5}$$

Lusztig introduced a certain modified version of the quantum group, denoted $\dot{\mathbf{U}}$, which has a canonical basis with the remarkable properties of positivity and integrality. To define $\dot{\mathbf{U}}$, we adjoin to $\mathbf{U}$ a system of projectors $\{1_n\}_{n \in \mathbb{Z}}$ such that $1_m 1_n = \delta_{mn} 1_m$ for all $m, n \in \mathbb{Z}$ and impose the relations

$$E 1_n = 1_{n+2} E, \quad F 1_n = 1_{n-2} F, \quad K^{\pm 1} 1_n = 1_n K^{\pm 1} = q^{\pm n} 1_n$$

Then $\dot{\mathbf{U}}$ is defined as the subalgebra (without unit) spanned by the elements $x 1_n, x \in \mathbf{U}, n \in \mathbb{Z}$.

1.1.2 Quasitriangular structure.

The quantum group $U_q(\mathfrak{sl}_2)$ possesses a quasitriangular structure encoded in the properties of the universal R-matrix [Dr]. Drinfeld's original description of the universal R-matrix required the introduction of a related quantum group, denoted $U_h(\mathfrak{sl}_2)$. The quantum group $U_h(\mathfrak{sl}_2)$ is an associative algebra over the ring $\mathbb{C}[[h]]$ of formal power series in a formal variable h. It has generators E, F, H satisfying the relations

$$HE - EH = 2E$$

$$HF - FH = -2F$$

$$EF - FE = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}$$

and comultiplication $\overline{\Delta}$ (Δ is similar)

$$\overline{\Delta}H = H \otimes 1 + 1 \otimes H$$

$$\overline{\Delta}E = E \otimes 1 + e^{-hH} \otimes E$$

$$\overline{\Delta}F = F \otimes e^{hH} + 1 \otimes F$$

Note that one can embed $U_q(\mathfrak{sl}_2)$ in $U_h(\mathfrak{sl}_2)$ by defining $K^{\pm 1} = e^{\pm hH}$ and $q^{\pm 1} = e^{\pm h}$. One can also realize $U_h(\mathfrak{sl}_2)$ in a completion of $U_q(\mathfrak{sl}_2)$ by formal power series $\mathbb{C}[[1-q]]$ so that $h = \log q \quad (= \log(1-(1-q)))$, $\quad H = \frac{\log K}{\log q}$.

The universal R-matrix has the form

$$\mathcal{R} = C\Theta \tag{1.6}$$

where

$$\begin{aligned} \Theta &= \sum_{n\geq 0} (-1)^n q^{-\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]!} F^n \otimes E^n \\ C &= \exp(-\frac{h}{2} H \otimes H) \end{aligned} \tag{1.7}$$

Then one has the following properties

$$\begin{aligned} \Theta\overline{\Delta}(x) &= \Delta(x)\Theta \\ C\Delta(x) &= \overline{\Delta}'(x)C \end{aligned} \tag{1.8}$$

where $\overline{\Delta}'$ is the opposite multiplication. These relations imply

$$\mathcal{R}\overline{\Delta}(x) = \overline{\Delta}'(x)\mathcal{R} \tag{1.9}$$

One can also check the quasitriangularity identities

$$\begin{aligned}(\overline{\Delta} \otimes 1)(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{23} \\ (1 \otimes \overline{\Delta})(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{12}\end{aligned} \tag{1.10}$$

The quasitriangularity identities imply the (universal) Yang-Baxter relation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{1.11}$$

This element belongs to a completion of $\mathbf{U}^{\otimes 3}$. We call it the full braiding and denote it by $\mathcal{R}^{(3)}$. Note that

$$\mathcal{R}^{(3)} = \mathcal{R}_{23}(1 \otimes \overline{\Delta})(\mathcal{R}) = \mathcal{R}_{12}(\overline{\Delta} \otimes 1)(\mathcal{R}) \tag{1.12}$$

The Yang-Baxter relation holds for Θ only in a modified form [L3], however the exact analogue of the identity (1.12) is still valid. Thus we can define

$$\Theta^{(3)} = \Theta_{23}(1 \otimes \overline{\Delta})(\Theta) = \Theta_{12}(\overline{\Delta} \otimes 1)(\Theta)$$

We also define

$$C^{(3)} = C_{23}(1 \otimes \overline{\Delta})(C) = C_{12}(\overline{\Delta} \otimes 1)(C)$$

It is easy to see that

$$\begin{aligned}(1 \otimes \overline{\Delta})(C) &= C_{12}C_{13} = C_{13}C_{12} \\ (\overline{\Delta} \otimes 1)(C) &= C_{23}C_{13} = C_{13}C_{23}\end{aligned}$$

We have a generalization of the factorization (1.6)

$$\mathcal{R}^{(3)} = C^{(3)}\Theta^{(3)} \tag{1.13}$$

In fact we have

$$\mathcal{R}^{(3)} = C_{12}\Theta_{12}(\overline{\Delta}\otimes 1)(C\Theta) = C_{12}\Theta_{12}(\overline{\Delta}\otimes 1)(C)(\overline{\Delta}\otimes 1)(\Theta) =$$

$$= C_{12}(\overline{\Delta}\otimes 1)(C)\Theta_{12}(\overline{\Delta}\otimes 1)(\Theta) = C^{(3)}\Theta^{(3)}$$

Here we use the commutativity of Θ_{12} and $(\overline{\Delta}\otimes 1)(C) = C_{23}C_{13}$, which can be verified as follows. First we note that

$$[1\otimes H\otimes H + H\otimes 1\otimes H, F^n\otimes E^n\otimes 1] = 0$$

It implies

$$\mathrm{ad}(-\frac{h}{2}1\otimes H\otimes H - \frac{h}{2}H\otimes 1\otimes H)\Theta_{12} = 0$$

Then exponentiation yields

$$C_{23}C_{13}\Theta_{12}C_{13}^{-1}C_{23}^{-1} = \exp(\mathrm{ad}(-\frac{h}{2}1\otimes H\otimes H - \frac{h}{2}H\otimes 1\otimes H))\Theta_{12} = \Theta_{12}$$

More generally, we define a full braiding $\mathcal{R}^{(n)}$ in a completion of $\mathbf{U}^{\otimes n}$

$$\mathcal{R}^{(n)} = (\mathcal{R}_{2,\ldots,n})^{(n-1)}(1\otimes\overline{\Delta}^{n-2})(\mathcal{R}) = (\mathcal{R}_{1,\ldots,n-1})^{(n-1)}(\overline{\Delta}^{n-2}\otimes 1)(\mathcal{R})$$

The elements $\Theta^{(n)}$ and $C^{(n)}$ are defined by analogous formulas and one has

PROPOSITION 1.1. $\mathcal{R}^{(n)} = C^{(n)}\Theta^{(n)}$

□

1.2. Category of finite-dimensional representations

1.2.1 The fundamental representation.

Let $V_1 = \mathbb{C}v_1 \oplus \mathbb{C}v_{-1}$ be a two-dimensional irreducible representation of $\mathbf{U}$ with the action defined by the formulas

$$K^{\pm 1}v_1 = q^{\pm 1}v_1, \quad K^{\pm 1}v_{-1} = q^{\mp 1}v_{-1}$$

$$Ev_1 = 0, \quad Ev_{-1} = v_1$$

$$Fv_1 = v_{-1}, \quad Fv_{-1} = 0$$

We define a bilinear symmetric pairing in V_1 by

$$< v_i, v_j >= \delta_{ij}, \quad i,j = \pm 1$$

One can easily check that

$$< xv_i, v_j >=< v_i, \omega(x)v_j >, \quad x \in \mathbf{U}$$

We call V_1 *the fundamental representation* of $\mathbf{U}$. We denote by $V_0 \cong \mathbb{C}$ the 1-dimensional representation of $\mathbf{U}$ given by the counit η.

1.2.2 Tensor powers of the fundamental representation.

Next we consider a tensor product of two fundamental representations $V_1 \otimes V_1$. We define the bilinear symmetric pairing in $V_1 \otimes V_1$ by

$$< v_i \otimes v_j, v_{j'} \otimes v_{i'} >= \delta_{ii'}\delta_{jj'}, \quad i, i', j, j' = \pm 1$$

Then we have for $x \in \mathbf{U}$

$$< \Delta(x)v_i \otimes v_j, v_{j'} \otimes v_{i'} >=< v_i \otimes v_j, \overline{\Delta}(\omega(x))v_{j'} \otimes v_{i'} >$$

Recall that we work primarily in the dual space with the dual action given by $\overline{\Delta}$, except for chapter III, where we use Δ when working with the canonical basis. To indicate the dual space we will use the upper indexes

$$v^1 = v_1, v^{-1} = v_{-1}$$

We introduce three intertwining operators with respect to the dual action given by $\overline{\Delta}$

$$\begin{aligned}
&\epsilon_1 : V_1 \otimes V_1 \to V_0 \\
&\epsilon_1(v^1 \otimes v^1) = \epsilon_1(v^{-1} \otimes v^{-1}) = 0 \\
&\epsilon_1(v^{-1} \otimes v^1) = 1, \quad \epsilon_1(v^1 \otimes v^{-1}) = -q \\
(1.14) \qquad &\delta_1 : V_0 \to V_1 \otimes V_1 \\
&\delta_1(1) = v^1 \otimes v^{-1} - q^{-1} v^{-1} \otimes v^1 \\
&\check{R}_{11} : V_1 \otimes V_1 \to V_1 \otimes V_1 \\
&\check{R}_{11} = P\mathcal{R}
\end{aligned}$$

Then we have

$$\begin{aligned}
&(\mathrm{Id}_{V_1} \otimes \epsilon_1) \circ (\delta_1 \otimes \mathrm{Id}_{V_1}) = \mathrm{Id}_{V_1} = (\epsilon_1 \otimes \mathrm{Id}_{V_1}) \circ (\mathrm{Id}_{V_1} \otimes \delta_1) \\
(1.15) \qquad &\epsilon_1 \circ \delta_1 = -q - q^{-1} \\
&\check{R}_{11} = q^{\frac{1}{2}} \delta_1 \circ \epsilon_1 + q^{-\frac{1}{2}} \mathrm{Id} \\
&\check{R}_{11}^2 = -(q - q^{-1}) q^{\frac{1}{2}} \check{R}_{11} + q \mathrm{Id}
\end{aligned}$$

Remark: Because of the appearance of the square root $q^{\frac{1}{2}}$, from now on we, strictly speaking, work over the field $\mathbb{C}(q^{\frac{1}{2}})$ of rational functions in $q^{\frac{1}{2}}$ rather than in q.

Now we consider the tensor product of n fundamental representations $V_1 \otimes ... \otimes V_1$ with the pairing

$$(1.16) \qquad < v_{i_1} \otimes ... \otimes v_{i_n}, v^{i'_n} \otimes ... \otimes v^{i'_1} >= \delta_{i_1}^{i'_1} ... \delta_{i_n}^{i'_n}$$

One can check that the action of $\Delta^{n-1}(x)$ is dual to the action of $\overline{\Delta}^{n-1}(x)$ for all $x \in \mathbf{U}$.

1.2.3 Temperley-Lieb algebra.

We recall that the Temperley-Lieb algebra TL_n is an algebra over $\mathbb{C}(q)$ with generators $U_1, ..., U_{n-1}$ and defining relations

$$
U_i^2 = -(q+q^{-1})U_i
$$
$$
U_i U_{i\pm 1} U_i = U_i \tag{1.17}
$$
$$
U_i U_j = U_j U_i, \quad |i-j| > 1
$$

This algebra admits a realization as the algebra of intertwining operators of the $U_q(\mathfrak{sl}_2)$-module $V_1^{\otimes n}$ as follows

$$
U_i \longrightarrow 1^{\otimes(i-1)} \otimes (\delta_1 \circ \epsilon_1) \otimes 1^{\otimes(n-i-1)} \tag{1.18}
$$

Moreover the Temperley-Lieb algebra TL_n is a factor algebra of the Hecke algebra H_n. The latter algebra is also defined over $\mathbb{C}(q)$ with generators $T_1, ..., T_{n-1}$ and defining relations

$$
T_i^2 = -(q-q^{-1})T_i + q
$$
$$
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{1.19}
$$
$$
T_i T_j = T_j T_i, \quad |i-j| > 1
$$

It admits a homomorphism to the algebra of intertwining operators of $V_1^{\otimes n}$ by the formula

$$
T_i \longrightarrow 1^{\otimes(i-1)} \otimes q^{-\frac{1}{2}} \check{R}_{11} \otimes 1^{\otimes(n-i-1)} \tag{1.20}
$$

The Temperley-Lieb algebra has a natural basis formed by reduced monomials in the generators $U_1, ..., U_{n-1}$. A monomial is called reduced if it cannot be transformed into another monomial with fewer factors. Also, equal reduced monomials are identified. We call the resulting basis *the dual canonical basis of the Temperley-Lieb algebra* TL_n and denote it by B_n^{TL}. The cardinality of B_n^{TL} is the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$. It follows from a q-version of Schur's duality [J] that the Temperley-Lieb algebra TL_n constitutes the whole algebra $\mathrm{End}_{\mathrm{U}}(V_1^{\otimes n})$.

1.2.4 Finite dimensional irreducible representations.

Tensor powers of the fundamental representation decompose into various direct sums of irreducible $(n+1)$-dimensional representations $V_n, n = 0, 1, 2, \ldots$. One can always choose a basis

$$\{v_m\}, \quad -n \leq m \leq n, \quad m = n (\mathrm{mod}\ 2)$$

of V_n such that the action of $\mathbf{U}$ is the following

$$K^{\pm 1} v_m = q^{\pm m} v_m$$
$$E v_m = [\frac{n+m}{2} + 1] v_{m+2}$$
$$F v_m = [\frac{n-m}{2} + 1] v_{m-2},$$

where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and $v_{n+2} = v_{-n-2} = 0$.

We define a bilinear symmetric pairing in V_n by the conditions

$$< xu, v > = < u, \omega(x) v >, \quad < v_n, v_n > = 1$$

where $u, v \in V_n$ and $x \in U$. This implies that

$$< v_{n-2k}, v_{n-2l} > = \delta_{k,l} \begin{bmatrix} n \\ k \end{bmatrix},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

Let $\{v^m\}_{m=-n}^{n}$ be the dual basis of $\{v_m\}_{m=-n}^{n}$ with respect to the form $<, >$. Clearly

(1.21)
$$v^{n-2k} = \begin{bmatrix} n \\ k \end{bmatrix}^{-1} v_{n-2k}$$

The action of $\mathbf{U}$ in the dual basis is the following:

(1.22)
$$K^{\pm 1} v^m = q^{\pm m} v_m$$
$$E v^m = [\frac{n-m}{2}] v^{m+2}$$
$$F v^m = [\frac{n+m}{2}] v^{m-2}$$

It is well-known that the irreducible representations $V_n, n = 0, 1, 2, \ldots$, constitute a complete set of so-called type I irreducible finite-dimensional representations, *i.e.*, representations such that K acts in every weight subspace by q^m, for some $m \in \mathbb{Z}$. The category of type I finite dimensional representations is closed under the tensor product. In particular, one has

$$V_m \otimes V_n \cong \bigoplus_{k=|m-n|}^{m+n} V_k.$$

where the sum is over those k such that $k - |m-n|$ is even.

1.2.5 Jones-Wenzl projectors.

Any irreducible representation V_n can be realized as a q-symmetric power inside $V_1^{\otimes n}$. We will write explicit formulas in the dual basis.

Let $s = (s_1, \ldots, s_n)$ and $s_i = \pm 1, 1 \leq i \leq n$. We denote

$$|s| = \sum_{i=1}^{n} s_i$$

and

$$\|s\|_+ = \sum_{i<j} \{s_i > s_j\}, \quad \|s\|_- = \sum_{i<j} \{s_i < s_j\} \tag{1.23}$$

where $\{a > b\} = 1$ if $a > b$, and 0 otherwise.

Proposition 1.2. *The following inclusions and projections intertwine the action of* $U_q(\mathfrak{sl}_2)$

$$\begin{aligned} &\iota_n : V_n \hookrightarrow V_1^{\otimes n} \\ &\iota_n(v^m) = \begin{bmatrix} n \\ \frac{n-m}{2} \end{bmatrix}^{-1} \sum_{s, |s|=m} q^{\|s\|_-} v^{s_1} \otimes \ldots \otimes v^{s_n} \\ &\pi_n : V_1^{\otimes n} \longmapsto V_n \\ &\pi_n(v^{s_1} \otimes \ldots \otimes v^{s_n}) = q^{-\|s\|_+} v^{|s|} \end{aligned} \tag{1.24}$$

The proof is straightforward. □

COROLLARY 1.3. *$\iota_n \circ \pi_n : V_1^{\otimes n} \to V_1^{\otimes n}$ is a projector, that is, $(\iota_n \circ \pi_n)^2 = \iota_n \circ \pi_n$.*

□

The operator $\iota_n \circ \pi_n$ is called *the Jones-Wenzl projector* and is also denoted p_n.

As we mentioned earlier, the Hecke algebra H_n has a natural homomorphism to the Temperley-Lieb algebra TL_n. To simplify notations, we identify the generator T_i pf the Hecke algebra H_n with its image in TL_n:

$$T_i = q^{-1} \cdot + U_i \tag{1.25}$$

The symmetric group $\mathbf{S}_n$ has a presentation by generators $s_1, ..., s_{n-1}$ and relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i, \quad |i-j| > 1.$$

For a permutation $s \in \mathbf{S}_n$ denote by $l(s)$ the number of pairs $(i,j), 1 \le i < j \le n, s(i) > s(j)$. A presentation of $s \in \mathbf{S}_n$ as a product $s_{i_1} ... s_{i_k}$ where $k = l(s)$ is called a *reduced representation* of s. For general s it is not unique.

For $s \in \mathbf{S}_n$ denote by $T(s)$ the element of the Temperley-Lieb algebra TL_n defined by

$$T(s) \stackrel{\text{def}}{=} T_{i_1} ... T_{i_k} \tag{1.26}$$

where $s_{i_1} ... s_{i_k}$ is a reduced representation of s and T_i is given by (1.25). It is easy to check that $T(s)$ does not depend on the choice of a reduced representation of s.

The following explicit formula for the Jones-Wenzl projector is proved in [KaL].

THEOREM 1.4. *The Jones-Wenzl projector*

$$p_n = \frac{1}{[n]_-!} \sum_{s \in \mathbf{S}_n} q^{-\frac{3l(s)}{2}} T(s) \tag{1.27}$$

where $[n]_-! := [1]_-[2]_- ... [n]_-$ and $[i]_- = \frac{q^{-2i}-1}{q^{-2}-1}$.

1.2.6 Duality intertwiners and braiding.

Next we introduce generalizations of three intertwining operators $\epsilon_1, \delta_1, \check{R}_{11}$ from the previous subsection:

$$
\begin{aligned}
&\epsilon_n : V_n \otimes V_n \to V_0 \\
&\epsilon_n = \epsilon_1 \circ (1 \otimes \epsilon_1 \otimes 1) \circ \ldots \circ (1^{\otimes(n-1)} \otimes \epsilon_1 \otimes 1^{\otimes(n-1)}) \circ (\iota_n \otimes \iota_n) \\
&\delta_n : V_0 \to V_n \otimes V_n \\
&\delta_n = (\pi_n \otimes \pi_n) \circ (1^{\otimes(n-1)} \otimes \delta_1 \otimes 1^{\otimes(n-1)}) \circ \ldots \circ (1 \otimes \epsilon_1 \otimes 1) \circ \delta_1 \\
&\check{R}_{mn} : V_m \otimes V_n \to V_n \otimes V_m \\
&\check{R}_{mn} = P\mathcal{R}
\end{aligned}
\tag{1.28}
$$

where P is the permutation $x \otimes y \to y \otimes x$.

The quasitriangular properties of $\mathcal{R}$ imply that

$$
\check{R}_{mn} = (\pi_n \otimes \pi_n) \circ \check{\check{R}}_{mn} \circ (\iota_m \otimes \iota_n),
\tag{1.29}
$$

where $\check{\check{R}}_{mn}$ is a composition of mn operators $\check{R}_{11}$ in the natural order. One can also derive generalizations of the identities (1.15).

1.3 Lusztig's canonical bases

1.3.1 Based modules.

We will briefly recall definition and properties of based modules introduced by Lusztig [L2]. We will use primarily Lusztig's notations. Let $\mathcal{A}$ denote $\mathbb{Z}[q, q^{-1}]$. We will consider finite dimensional **U**-modules of type I. For any such module M one has a decomposition $M = \bigoplus_{\lambda \in \mathbb{Z}} M^\lambda$, where

$$M^\lambda = \{m \in M | Km = q^\lambda m\}.$$

Let B be a $\mathbb{C}(q)$-basis of M. We define an involution $\sigma_B : M \to M$ by

$$\sigma_B(fb) = \overline{f}b \tag{1.30}$$

for all $f \in \mathbb{C}(q)$ and all $b \in B$. Then (M, B) is called a *based module* (with respect to the choice of generators $E, F, K^{\pm 1}$ of $\mathbf{U}$) if the following conditions are satisfied:

(a) $B \cap M^\lambda$ is a basis of M^λ, for any $\lambda \in \mathbb{Z}$.

(b) The $\mathcal{A}$-submodule ${}_{\mathcal{A}}M$ generated by B is stable under $E^n/[n]!$ and $F^n/[n]!$.

(c) The involution σ_B is compatible with the involution σ on $\mathbf{U}$ in the sense that

$$\sigma_B(um) = \sigma(u)\sigma_B(m) \tag{1.31}$$

for all $u \in \mathbf{U}, m \in M$.

(d) B is a crystal basis of M at ∞.

The notion of a crystal basis was introduced by Kashiwara [Ks]. For our purposes we only need the fact that $\{v_m\}_{m=-n}^{n}$ is a crystal basis of V_n at ∞ for all $n \in \mathbb{N}$.

The direct sum of two based modules (M, B) and (M', B') is again a based module $(M \oplus M', B \cup B')$.

The tensor product of two based modules $M \otimes M'$ with the obvious basis $B \otimes B'$ does not in general satisfy property (c) of the definition. Lusztig introduces a modified basis $B \diamondsuit B'$ in the tensor product as follows

Let $\Psi : M \otimes M' \to M \otimes M'$ be given by

$$\Psi(m \otimes m') = \Theta(\sigma_B(m) \otimes \sigma_{B'}(m')) \tag{1.32}$$

Then

$$\Psi^2 = 1 \tag{1.33}$$

because $\Theta\overline{\Theta} = 1 \otimes 1$. Also

$$\Psi(u(m \otimes m')) = \sigma(u)\Psi(m \otimes m'), \quad u \in \mathbf{U}$$

The involution Ψ will be the associated involution $\sigma_{B\diamondsuit B'}$ for the modified basis. Let ${}_{\mathcal{A}}M\otimes M'$ (respectively ${}_{\mathbb{Z}[q^{-1}]}M\otimes M'$) be the $\mathcal{A}$-submodule (respectively $\mathbb{Z}[q^{-1}]$ submodule) of $M\otimes M'$ generated by the basis $B\otimes B'$. The set $B\times B'$ has a partial ordering such that $(b_1,b_1')\geq(b_2,b_2')$ if and only if

$$b_1\in M^{\lambda_1},b_1'\in M'^{\lambda_1'},b_2\in M^{\lambda_2},b_2'\in M'^{\lambda_2'},$$

$$\lambda_1\geq\lambda_2,\lambda_1'\leq\lambda_2',\lambda_1+\lambda_1'=\lambda_2+\lambda_2'$$

Then Lusztig proves the following:

THEOREM 1.5.

(a) For any $(b_1,b_1')\in B\times B'$, there is a unique element $b_1\diamondsuit b_1'\in_{\mathbb{Z}[q^{-1}]} M\otimes M'$ such that

$$\Psi(b_1\diamondsuit b_1')=b_1\diamondsuit b_1'$$

and

$$b_1\diamondsuit b_1'-b_1\otimes b_1'\in q^{-1}\cdot_{\mathbb{Z}[q^{-1}]} M\otimes M'$$

(b) The element $b_1\diamondsuit b_1'$ in (a) is equal to $b_1\otimes b_1'$ plus a linear combination of elements $b_2\otimes b_2'$ with

$$(b_2,b_2')\in B\times B',(b_2,b_2')<(b_1,b_1')$$

and with coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$.

(c) The elements $b_1\diamondsuit b_1'$ with b_1,b_1' as above, form a $\mathbb{C}(q)$-basis $B\diamondsuit B'$ of $M\otimes M'$, an $\mathcal{A}$-basis of ${}_{\mathcal{A}}M\otimes M'$ and a $\mathbb{Z}[q^{-1}]$-basis of ${}_{\mathbb{Z}[q^{-1}]}M\otimes M'$.

This theorem implies that $(M\otimes M',B\diamondsuit B')$ is a based module with associated involution Ψ.

1.3.2 Canonical basis for the tensor product of two irreducible representations.

The first class of examples of based modules for $\mathbf{U}$ is provided by irreducible representations and canonical basis

$$(V_n,\{v_m\}_{m=-n}^{n}),n\in\mathbb{Z}_+$$

Lusztig also gives an example of based modules associated with the tensor product of two irreducible representations

$$(V_m \otimes V_n, \{v_k \diamondsuit v_l\}), m, n \in \mathbb{Z}_+, -m \leq k \leq m, -n \leq l \leq n$$

One has

$$v_{m-2k} \diamondsuit v_{2l-n} = \sum_{s \geq 0} q^{-s(s+k)} \begin{bmatrix} s+l \\ s \end{bmatrix} v_{m-2k-2s} \otimes v_{2l-n+2s}, \quad k \geq l \geq 0$$

$$v_{m-2k} \diamondsuit v_{2l-n} = \sum_{s \geq 0} q^{-s(s+l)} \begin{bmatrix} s+k \\ s \end{bmatrix} v_{m-2k-2s} \otimes v_{2l-n+2s}, \quad l \geq k \geq 0$$

One can also derive the inverse relations

$$v_{m-2k} \otimes v_{2l-n} = \sum_{s \geq 0} (-1)^s q^{-s(k+1)} \begin{bmatrix} s+l \\ s \end{bmatrix} v_{m-2k-2s} \diamondsuit v_{2l-n+2s}, \quad k \geq l \geq 0$$

$$v_{m-2k} \otimes v_{2l-n} = \sum_{s \geq 0} (-1)^s q^{-s(l+1)} \begin{bmatrix} s+k \\ s \end{bmatrix} v_{m-2k-2s} \diamondsuit v_{2l-n+2s}, \quad l \geq k \geq 0$$

Finally one can compute the action of the generators of $\mathbf{U}$

$$E(v_{m-2k} \diamondsuit v_{2l-n}) = [m-k+1](v_{m-2k+2} \diamondsuit v_{2l-n}) +$$

$$\{l+1 > k\}[m-2k+l+1](v_{m-2k} \diamondsuit v_{2l-n+2})$$

$$F(v_{m-2k} \diamondsuit v_{2l-n}) = [n-l+1](v_{m-2k} \diamondsuit v_{2l-n-2}) +$$

$$\{k+1 > l\}[n-2l+k+1](v_{m-2k-2} \diamondsuit v_{2l-n})$$

where $\{l+1 > k\} = 1$ if $l+1 > k$ and 0 otherwise.

We will rewrite these formulas in the dual basis since it will be more suitable for the graphical interpretation.

1.3.3 Basis dual to Lusztig's canonical basis: example of a tensor product of two irreducibles.

Let us define the bilinear pairing $<,>$ of $V_m \otimes V_n$ and $V_n \otimes V_m$ as the product of the corresponding bilinear pairing in the factors, i.e.

$$< v_k \otimes v_l, v^{l'} \otimes v^{k'} >= \delta_k^{k'} \delta_l^{l'}$$

Then we have for $x \in \mathbf{U}$

$$< \Delta(x)v_k \otimes v_l, v^{l'} \otimes v^{k'} >=< v_k \otimes v_l, \overline{\Delta}(\omega(x))v^{l'} \otimes v^{k'} >$$

Thus the natural action of $\mathbf{U}$ in the dual tensor product is given by the comultiplication $\overline{\Delta}$.

We define the dual of Lusztig's basis in a tensor product with respect to the form $<,>$:

$$< v_k \diamondsuit v_l, v^{l'} \heartsuit v^{k'} >= \delta_k^{k'} \delta_l^{l'}$$

The above formulas imply

PROPOSITION 1.6. *Explicit expressions for the dual basis in the tensor product $V_n \otimes V_m$ and the action of $\mathbf{U}$ in this basis are the following*

$$(i) \quad v^{2l-n} \otimes v^{m-2k} = \sum_{s\geq 0} q^{-sk} \begin{bmatrix} l \\ s \end{bmatrix} v^{2l-n-2s} \heartsuit v^{m-2k+2s}, \quad k \geq l \geq 0$$

$$v^{2l-n} \otimes v^{m-2k} = \sum_{s\geq 0} q^{-sl} \begin{bmatrix} k \\ s \end{bmatrix} v^{2l-n-2s} \heartsuit v^{m-2k+2s}, \quad l \geq k \geq 0$$

$$(ii) \quad v^{2l-n} \heartsuit v^{m-2k} = \sum_{s\geq 0} (-1)^s q^{-s(k-s+1)} \begin{bmatrix} l \\ s \end{bmatrix} v^{2l-n-2s} \otimes v^{m-2k+2s}, \quad k \geq l \geq 0$$

$$v^{2l-n} \heartsuit v^{m-2k} = \sum_{s\geq 0} (-1)^s q^{-s(l-s+1)} \begin{bmatrix} k \\ s \end{bmatrix} v^{2l-n-2s} \otimes v^{m-2k+2s}, \quad l \geq k \geq 0$$

$$(iii) \quad E(v^{2l-n} \heartsuit v^{m-2k}) = [n-l] v^{2l-n+2} \heartsuit v^{m-2k} +$$

$$\{k > l\}[n - 2l + k] v^{2l-n} \heartsuit v^{m-2k+2}$$

$$F(v^{2l-n} \heartsuit v^{m-2k}) = [m-k] v^{2l-n} \heartsuit v^{m-2k-2} +$$

$$\{l > k\}[m - 2k + l] v^{2l-n-2} \heartsuit v^{m-2k}$$

We will have a simple graphical interpretation of formulas (ii) and (iii) in Section 2.3.

1.3.4 Canonical and dual canonical bases in tensor product of finite number of irreducible representations.

Repeatedly applying the Lusztig construction, one obtains a canonical basis in the tensor product of r irreducible representations $V_{a_1} \otimes ... \otimes V_{a_n}$. Moreover, Lusztig proves the associativity of the tensor product construction that guarantees that the canonical basis

$$\{v_{k_1} \diamondsuit ... \diamondsuit v_{k_n}\}, \quad -a_i \leq k_i \leq a_i$$

is independent of parenthesization. The associated involution $\Psi^{(n)}$ is defined by

$$\Psi^{(n)} = \Theta^{(n)} \circ \sigma_{B_{a_1}} \otimes ... \otimes \sigma_{B_{a_n}}, \tag{1.34}$$

where B_{a_i} denotes the canonical basis in $V_{a_i}, i = 1, ..., n$, and $\Theta^{(n)}$ by abuse of notation denotes the operator of multiplication by $\Theta^{(n)}$ introduced in Section 1.1.2. We also define $\overline{\Psi}^{(n)}$ by the same formula with $\Theta^{(n)}$ replaced by $\overline{\Theta}^{(n)}$.

Lusztig's theorem implies that the canonical basis can be characterized by the two conditions:

$$\Psi^{(n)}(v_{k_1} \diamondsuit ... \diamondsuit v_{k_n}) = v_{k_1} \diamondsuit ... \diamondsuit v_{k_n}$$

$$v_{k_1} \diamondsuit ... \diamondsuit v_{k_n} - v_{k_1} \otimes ... \otimes v_{k_n} \in q^{-1} \cdot_{\mathbb{Z}[q^{-1}]} V_{a_1} \otimes ... \otimes V_{a_n}$$

In fact one can proceed by induction. For $n = 2$ it is a special case of Theorem 1.5(a). In the inductive step, we apply Theorem 1.5(a) to $M = V_{a_1} \otimes ... \otimes V_{a_{n-1}}$ and $M' = V_{a_n}$ with bases $B_{a_1} \diamondsuit ... \diamondsuit B_{a_{n-1}}$ and B_{a_n}, respectively. One has the associated involution

$$(\Delta^{(n-2)} \otimes 1)(\Theta) \circ (\Psi^{(n-1)} \otimes \sigma_{B_{a_n}}) =$$

$$(\Delta^{(n-2)} \otimes 1)(\Theta) \circ (\Theta^{(n-1)} \otimes 1) \circ (\sigma_{B_{a_1}} \otimes ... \otimes \sigma_{B_{a_n}}) = \Psi^{(n)}$$

Theorem 1.5(a) also guarantees the existence of a unique element

$$x = (v_{k_1} \diamondsuit ... \diamondsuit v_{k_{n-1}}) \diamondsuit v_{k_n}$$

such that

$$x - (v_{k_1} \diamondsuit ... \diamondsuit v_{k_{n-1}}) \otimes v_{k_n} \in q^{-1}_{\mathbb{Z}[q^{-1}]} (V_{a_1} \otimes ... \otimes V_{a_{n-1}}) \otimes V_{a_n}.$$

Thus to show that x is also the unique element such that

$$x - v_{k_1} \otimes ... \otimes v_{k_n} \in q^{-1}_{\mathbb{Z}[q^{-1}]} V_{a_1} \otimes ... \otimes V_{a_{n-1}} \otimes V_{a_n}$$

we will use the induction assumption and the fact that the bases $B_{a_1} \diamondsuit ... \diamondsuit B_{a_{n-1}}$ and $B_{a_1} \otimes ... \otimes B_{a_{n-1}}$ are related by a triangular matrix with 1's on diagonal and off diagonal elements from $q^{-1}\mathbb{Z}[q^{-1}]$. For two factors ($n = 3$) this immediately follows from Theorem 1.5(b). For general n it is again proved by induction by repeatedly applying Theorem 1.5(b).

For convenience in the graphical interpretation we switch to the dual basis. We define the bilinear pairing $<,>$ of $V_{a_1} \otimes ... \otimes V_{a_n}$ and $V_{a_n} \otimes ... \otimes V_{a_1}$ by

$$< v_{k_1} \otimes ... \otimes v_{k_n}, v^{k'_n} \otimes ... \otimes v^{k'_1} >= \delta^{k'_1}_{k_1} ... \delta^{k'_n}_{k_n}$$

As in the previous subsection, the action of $\mathbf{U}$ in the dual basis in given by $\overline{\Delta}^{(n-1)}$.

We also define the dual of Lusztig's basis with respect to this form

$$< v_{k_1} \diamondsuit ... \diamondsuit v_{k_n}, v^{k'_n} \heartsuit ... \heartsuit v^{k'_1} >= \delta^{k'_1}_{k_1} ... \delta^{k'_n}_{k_n}$$

We denote by B^{a_i} the dual canonical basis $\{v^{k_i}\}, -a_i \leq k_i \leq a_i, k_i = a_i (\bmod 2)$, of V_{a_i} and by $B^{a_n} \heartsuit ... \heartsuit B^{a_1}$ the dual of Lusztig's canonical basis of $V_{a_1} \otimes ... \otimes V_{a_n}$.

Then the above argument implies the following

THEOREM 1.7.

(a) For any $-a_i \leq k_i \leq a_i, i = 1, ..., n, k_i = a_i (\mathrm{mod}\ 2)$ there exists a unique element

(1.35) $$v^{k_n} \heartsuit ... \heartsuit v^{k_1} \in V_{a_n} \otimes ... \otimes V_{a_1}$$

such that

(1.36) $$\overline{\Psi}^{(n)} (v^{k_n} \heartsuit ... \heartsuit v^{k_1}) = v^{k_n} \heartsuit ... \heartsuit v^{k_1}$$

and

$$(1.37) \qquad v^{k_n}\heartsuit...\heartsuit v^{k_1} - v^{k_n}\otimes ... \otimes v^{k_1} \in q^{-1}\ {}_{\mathbb{Z}[q^{-1}]}V_{a_n}\otimes ... \otimes V_{a_1}$$

(b) The element $v^{k_n}\heartsuit...\heartsuit v^{k_1}$ in (a) is equal to $v^{k_n}\otimes ... \otimes v^{k_1}$ plus a linear combination of elements $v^{l_n}\otimes ... \otimes v^{l_1}$ with $\sum_{i=1}^n l_i = \sum_{i=1}^n k_i$ and $\sum_{i=1}^{n'} l_i \geq \sum_{i=1}^{n'} k_i$ for all $1 \leq n' \leq n, (l_n, ..., l_1) \neq (k_n, ..., k_1)$ and with coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$.

(c) The elements $v^{k_n}\heartsuit...\heartsuit v^{k_1}$ form a $\mathbb{C}(q)$-basis $B^{a_n}\heartsuit...\heartsuit B^{a_1}$ of $V_{a_n}\otimes ... \otimes V_{a_1}$, an $\mathcal{A}$-basis of ${}_{\mathcal{A}}V_{a_n}\otimes ... \otimes V_{a_1}$ and a $\mathbb{Z}[q^{-1}]$-basis of ${}_{\mathbb{Z}[q^{-1}]}V_{a_n}\otimes ... \otimes V_{a_1}$.

Proof: The proof of the corresponding dual statement is explained above. To complete the proof we need to transfer it to the dual picture. We will use the following identity

$$(1.38) \quad < \Theta^{(n)} v_{k_1}\otimes ... \otimes v_{k_n}, v^{k'_n}\otimes ... \otimes v^{k'_1} > = < v_{k_1}\otimes ... \otimes v_{k_n}, \Theta^{(n)} v^{k'_n}\otimes ... \otimes v^{k'_1} >$$

In fact for $n = 2$ it follows from the definition of Θ and the form $<,>$.

For $n = 3$ we have

$$< (\Delta\otimes 1)(F^{(m)}\otimes E^{(m)})(\Theta\otimes 1)v_{k_1}\otimes v_{k_2}\otimes v_{k_3}, v^{k'_3}\otimes v^{k'_2}\otimes v^{k'_1} > =$$

$$< v_{k_1}\otimes v_{k_2}\otimes v_{k_3}, (1\otimes\Theta)(1\otimes\overline{\Delta})(F^{(m)}\otimes E^{(m)})v^{k'_3}\otimes v^{k'_2}\otimes v^{k'_1} > =$$

$$< v_{k_1}\otimes v_{k_2}\otimes v_{k_3}, (1\otimes\Delta)(F^{(m)}\otimes E^{(m)})(1\otimes\Theta)v^{k'_3}\otimes v^{k'_2}\otimes v^{k'_1} >$$

For general n the inductive step is similar.

Then (a) follows from

$$(1.39) \qquad \Theta^{(n)}\sigma_1\otimes ... \otimes\sigma_n = \sigma_1\otimes ... \otimes\sigma_n\overline{\Theta}^{(n)}$$

where we abbreviated σ_{V_i} to $\sigma_i, i = 1, ..., n$.

The other assertions immediately follow from the duality. □

1.4 Explicit formulas for canonical and dual canonical bases in arbitrary tensor products

1.4.1 Dual canonical basis.

One of our main results is an explicit form for the dual canonical basis in a tensor product $V_{a_n} \otimes ... \otimes V_{a_1}$ of finitely many irreducible representations. We will also identify the subset of the dual canonical basis which spans the subspace of U-invariants, denoted $\mathrm{Inv}_{\mathbf{U}}(V_{a_n} \otimes ... \otimes V_{a_1})$.

First we will describe the dual canonical basis of $V_1^{\otimes n}$. Denote by $1^{\otimes k}$ the identity operator $V_1^{\otimes k} \to V_1^{\otimes k}$.

THEOREM 1.8. *The following 3 rules provide a recursive construction of dual canonical vectors* $v_{\epsilon_n} \heartsuit ... \heartsuit v_{\epsilon_1} \in V_1^{\otimes n}$, *where* $\epsilon_1, ..., \epsilon_n \in \{1, -1\}$*:*

$$(i) \quad v_{-1} \heartsuit v_{\epsilon_{n-1}} \heartsuit ... \heartsuit v_{\epsilon_1} = v_{-1} \otimes (v_{\epsilon_{n-1}} \heartsuit ... \heartsuit v_{\epsilon_1})$$

$$(ii) \quad v_{\epsilon_n} \heartsuit ... \heartsuit v_{\epsilon_2} \heartsuit v_1 = (v_{\epsilon_n} \heartsuit ... \heartsuit v_{\epsilon_2}) \otimes v_1$$

$$(iii) \quad v_{\epsilon_n} \heartsuit ... \heartsuit v_{\epsilon_i} \heartsuit v_1 \heartsuit v_{-1} \heartsuit v_{\epsilon_{i-3}} \heartsuit ... \heartsuit v_{\epsilon_1} =$$

$$(1^{\otimes (n-i+1)} \otimes \delta_1 \otimes 1^{\otimes (i-3)}) v_{\epsilon_n} \heartsuit ... \heartsuit v_{\epsilon_i} \heartsuit v_{\epsilon_{i-3}} \heartsuit ... \heartsuit v_{\epsilon_1}$$

Proof: Claims (i) and (ii) of the theorem are trivial, for (iii) we need to check two conditions of Theorem 1.7(a). Of course, the second condition immediately follows from the formula (1.14) for δ_1, the first condition, *i.e.,* the invariance of the elements of the dual canonical basis with respect to $\overline{\Psi}^{(n)}$ will be verified in Section 2.4 using graphical calculus. □

Next we give a general result about the form of the dual canonical basis.

Given $k_i, -a_i \leq k_i \leq a_i, k_i = a_i (\mathrm{mod}\ 2), 1 \leq i \leq n$, we want to have an explicit formula for the dual canonical vector

$$v^{a_n - 2k_n} \heartsuit ... \heartsuit v^{a_1 - 2k_1} \in V_{a_n} \otimes ... \otimes V_{a_1}$$

Let $(\epsilon_a, ..., \epsilon_1)$, where $a = a_1 + ... + a_n, \epsilon_i \in \{1, -1\}$, be a sequence of ones and negative ones that starts with k_n copies of negative one, followed by $a_n - k_n$ copies of one, followed by k_{n-1} copies of negative one, followed by $a_{n-1}-k_{n-1}$ copies of one, ..., and ending with a_1-k_1 copies of one. Then we have

THEOREM 1.9.

$$v^{a_n-2k_n}\heartsuit\ldots\heartsuit v^{a_1-2k_1}=(\pi_{a_n}\otimes\ldots\otimes\pi_{a_1})v_{\epsilon_a}\heartsuit\ldots\heartsuit v_{\epsilon_1},$$

where $v^{a_n-2k_n}\heartsuit\ldots\heartsuit v^{a_1-2k_1}$ is a dual canonical vector of $V_{a_n}\otimes\ldots\otimes V_{a_1}$ and $v_{\epsilon_a}\heartsuit\ldots\heartsuit v_{\epsilon_1}$ a dual canonical vector of $V_1^{\otimes a}$ with $\epsilon_1,\ldots,\epsilon_a$ as defined above.

Proof: We need again to check the two conditions of Theorem 1.7(a). Again, the second condition is obvious from the formula (1.14), while the invariance with respect to $\overline{\Psi}^{(n)}$ will be demonstrated in Section 2.4. □

COROLLARY 1.10. *Under the isomorphism $End_{\mathrm{U}}(V_1^{\otimes n})\cong Inv_{\mathrm{U}}(V_1^{\otimes(2n)})$ given by*

$$S\to(S\otimes 1^{\otimes n})\circ(1^{\otimes(n-1)}\otimes\delta_1\otimes 1^{\otimes(n-1)})\circ\ldots\circ(1\otimes\delta_1\otimes 1)\circ\delta_1 \tag{1.42}$$

(where $S\in End_{\mathrm{U}}(V_1^{\otimes n})$) one has a bijection of the dual canonical basis B_n^{TL} of the Temperley-Lieb algebra and the dual canonical basis in $Inv_{\mathrm{U}}(V_1^{\otimes(2n)})$.

1.4.2 Canonical basis in tensor products.

Denote by $v(x_1,y_1;\ldots;x_k,y_k)$ the element of the canonical basis of $V_1^{\otimes n}, n=\sum_{i=1}^k x_i+y_i$ with the lexicographically highest term $v_1^{\otimes x_1}\otimes v_{-1}^{\otimes y_1}\otimes\ldots\otimes v_1^{\otimes x_k}\otimes v_{-1}^{\otimes y_k}$. We allow some of x_i and y_i to be 0. Thus,

$$v(x_1,y_1;\ldots;x_k,y_k)=v_1^{\otimes x_1}\diamondsuit v_{-1}^{\otimes y_1}\diamondsuit\ldots\diamondsuit v_1^{\otimes x_k}\diamondsuit v_{-1}^{\otimes y_k}. \tag{1.43}$$

LEMMA 1.11.

(i) $v(0,y_1;x_2,\ldots;x_k,y_k)=v_{-1}^{\otimes y_1}\otimes v(x_2,y_2,\ldots,x_k,y_k)$,

(ii) $v(x_1,y_1;\ldots,y_{k-1};x_k,0)=v(x_1,y_1,\ldots,x_{k-1},y_{k-1})\otimes v_1^{x_k}$.

The lemma holds because $v_{-1}^{\otimes y_1}$ is the lexicographically lowest element of the canonical basis of $V^{\otimes y_1}$ and so, if b is an element of the canonical basis of an arbitrary tensor product of finite-dimensional modules, we have $v_{-1}^{\otimes y_1} \diamondsuit b = v_{-1}^{\otimes y_1} \otimes b$. A similar argument implies part (*ii*) of the lemma. □

Recall that we denote by $1^{\otimes m}$ the identity operator $V_1^{\otimes m} \to V_1^{\otimes m}$.

THEOREM 1.12. *(i) If $y_{i-1} \geq x_i$ and $y_i \leq x_{i+1}$ then*

$$v(x_1, y_1; ...; x_k, y_k) = \begin{bmatrix} x_i + y_i \\ x_i \end{bmatrix} (1^{\otimes l} \otimes p_{x_i+y_i} \otimes 1^{\otimes j})$$
$$v(x_1, y_1; ...; x_{i-1}, y_{i-1} + y_i; x_i + x_{i+1}, y_{i+1}; ...; x_k, y_k)$$

where $l = \sum_{t<i} x_t + y_t$, $j = \sum_{t>i} x_t + y_t$.

(ii) If $y_1 \leq x_2$ then

$$v(x_1, y_1; ...; x_k, y_k) =$$
$$= \begin{bmatrix} x_1 + y_1 \\ x_1 \end{bmatrix} (p_{x_1+y_1} \otimes 1^{n-x_1-y_1})(v_{-1}^{\otimes y_1} \otimes v(x_1 + x_2, y_2; ...; x_k, y_k)),$$

(iii) If $y_{k-1} \geq x_k$ then

$$v(x_1, y_1; ...; x_k, y_k) =$$
$$= \begin{bmatrix} x_k + y_k \\ x_k \end{bmatrix} (1^{\otimes n-x_k-y_k} \otimes p_{x_k+y_k})(v(x_1, y_1; ...; x_{k-1}, y_{k-1} + y_k) \otimes v_1^{\otimes x_k}).$$

Applying this theorem k times, each time to the canonical basis vector that appears on the RHS, we can get an explicit formula for any canonical basis vector $v(x_1, y_1; ...; x_k, y_k)$. Thus, the theorem explicitly describes all vectors in the canonical basis of the tensor product $V_1^{\otimes n}$. The proof of this theorem will be given in Section 3.2.

It is easy to extract the canonical basis in an artitrary tensor product from the canonical basis of a tensor power of the fundamental representation because of the following theorem.

THEOREM 1.13. *The element* $v_{a_1-2k_1}\diamondsuit\ldots\diamondsuit v_{a_n-2k_n}$ *of the canonical basis of a tensor product* $V_{a_1}\otimes\ldots\otimes V_{a_n}$ *is given by*

$$v_{a_1-2k_1}\diamondsuit\ldots\diamondsuit v_{a_n-2k_n} = (\pi_{a_1}\otimes\ldots\otimes\pi_{a_n})v(a_1-k_1,k_1;\ldots;a_n-k_n,k_n).$$

Thus, the canonical basis of the tensor product $V_{a_1}\otimes\ldots\otimes V_{a_n}$ is obtained by projecting certain canonical basis vectors of the tensor product $V_1^{\otimes(a_1+\ldots+a_n)}$.

CHAPTER II

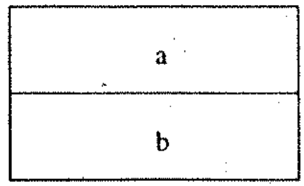
GRAPHICAL CALCULUS AND DUAL CANONICAL BASIS

In Sections 2.1 and 2.2 we recall the graphical calculus of **U**-intertwiners and slightly generalize it, by introducing oriented arcs, to describe the matrix coefficients of intertwiners. A similar generalization was independently introduced in [CFS]. Also, in the non-quantized case ($q = 1$) physicists had long been using arrows to depict weight vectors of $\mathfrak{sl}_2$ representations (see [M], for instance). In Section 2.3 we describe the dual canonical basis graphically and in Section 2.4 prove the validity of our description.

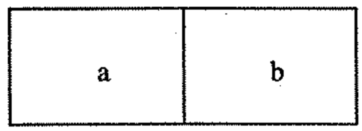
2.1. Temperley-Lieb algebra

Recall that throughout this section the action of $U_q(\mathfrak{sl}_2)$ in tensor products is given by the dual comultiplication $\overline{\Delta}$ and its iterations.

We will depict diagrams corresponding to intertwining operators $a : V_1^{\otimes m} \to V_1^{\otimes n}$ by certain curves connecting m distinct points on one horizontal line and n distinct points on another horisontal line lying above the first one. Only simple intersections are allowed. At each intersection we specify the type of intersection: over-crossing or under-crossing. Thus, any diagram can be viewed as a projection of a system of curves in three dimensions. The curves will usually be inside the box bounded by the two horizontal lines and the vertical lines determined by the extreme left and right points. Composition of two intertwining operators is obtained by concatenating two boxes vertically from below.

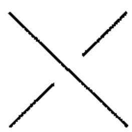
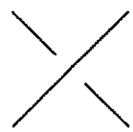
a ∘ b 

The tensor product of two intertwining operators corresponds to concatenating two boxes horizontally

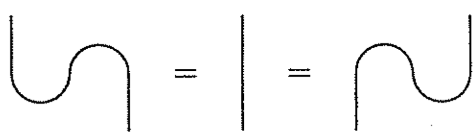
a ⊗ b 

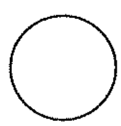
The identity map $I_n : V_1^{\otimes n} \to V_1^{\otimes n}$ is given by n parallel vertical lines. The other basic intertwining operators correspond to the following diagrams

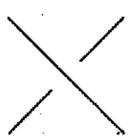
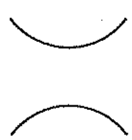
ϵ_1  δ_1 

$\check{R}_{11}$  $\check{R}_{11}^{-1}$ 

Then the relations (1.15) can be expressed as follows



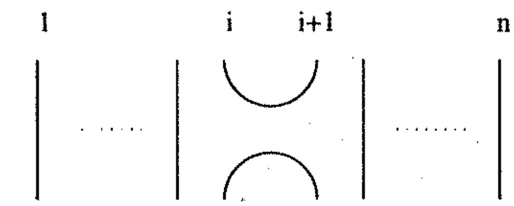
 $= -q - q^{-1}$

 $= q^{\frac{1}{2}}$  $+ q^{-\frac{1}{2}}$

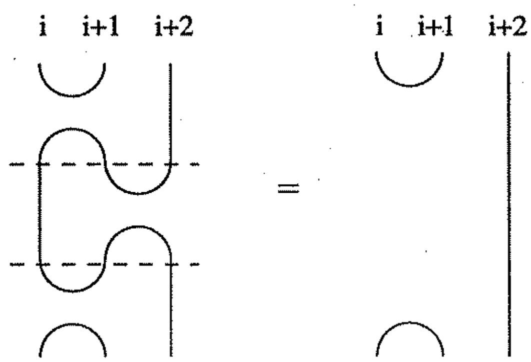
To describe the Temperley-Lieb algebra TL_n, one can consider diagrams of n simple, pairwise disjoint arcs in $\mathbb{R} \times [0,1]$ connecting n points on $\mathbb{R} \times \{0\}$ (say, the points $(1,0), ..., (n,0)$) with n points on $R \times \{1\}$ (the points $(1,1), ..., (n,1)$).

The multiplication is given by the composition as above combined with rescaling to bring the endpoints of the composition to the positions fixed as above.

The generator U_i of the Temperley-Lieb algebra has the following diagram

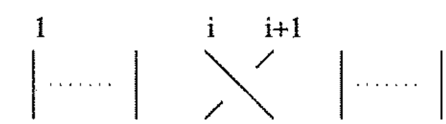


and the second defining relation can be visualized as an isotopy

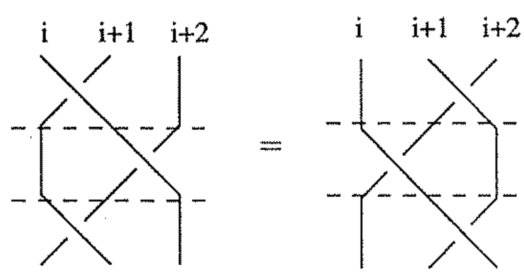


or its mirror image. The isotopy classes of diagrams in the Temperley-Lieb algebra TL_n provide a graphical realization of the dual canonical basis B_n^{TL} introduced in Section 1.2.3.

The generator T_i of the Hecke algebra is given by the diagram



multiplied by $q^{-\frac{1}{2}}$ and the second defining relation has the form of the braid relation

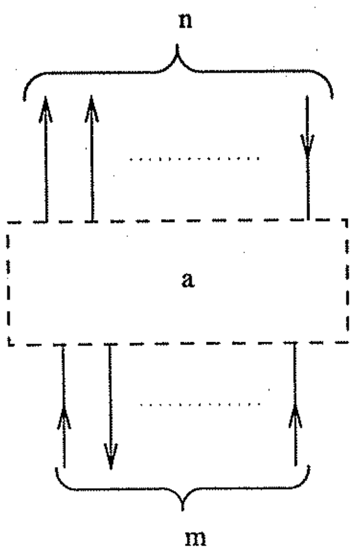


Given a diagram a with m bottom and n top ends, by abuse of notations we denote by the same letter a the intertwiner $V_1^{\otimes m} \to V_1^{\otimes n}$ associated to the diagram a.

A matrix coefficient

$$< a(v^{s_1} \otimes ... \otimes v^{s_m}), v^{t_n} \otimes ... \otimes v^{t_1} >$$

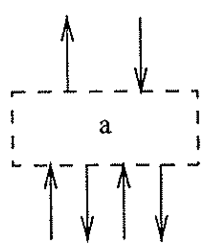
(where $s_i, t_j = \pm 1, i = 1, ..., m, j = 1, ..., n$) of $a : V_1^{\otimes m} \to V_1^{\otimes n}$ will be depicted by the diagram



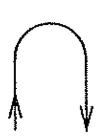
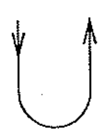
where we denote

$$v^1 \text{ by } \uparrow \quad and \quad v^{-1} \text{ by } \downarrow$$

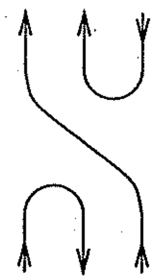
For example, the coefficient $< a(v^1 \otimes v^{-1} \otimes v^1 \otimes v^{-1}), v^{-1} \otimes v^1 >$ is depicted by



If a is presented by a set of disjoint arcs then the matrix coefficient is zero unless the orientations of two ends of each arcs are compatible. Each arc with a consistent orientation of ends contributes a factor 1 except in the following two cases

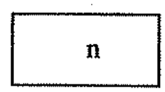
 $= -q$  $= -q^{-1}$

For instance, the value of the diagram below is $-q$.

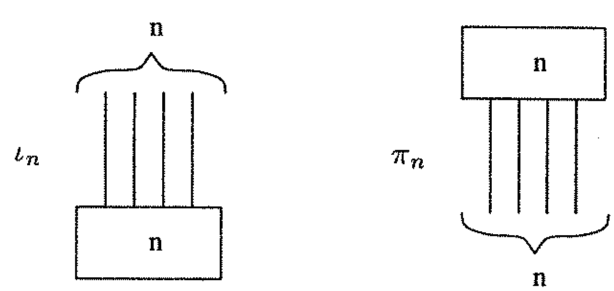


2.2. Representation category

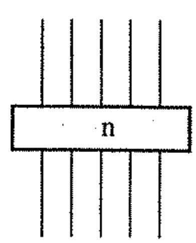
The irreducible representation V_n is depicted by a box marked by n.



The injector ι_n and the projector π_n are depicted by n lines exiting (resp. entering) a box marked by n. Sometimes, when it is clear what the marking is, we omit it.



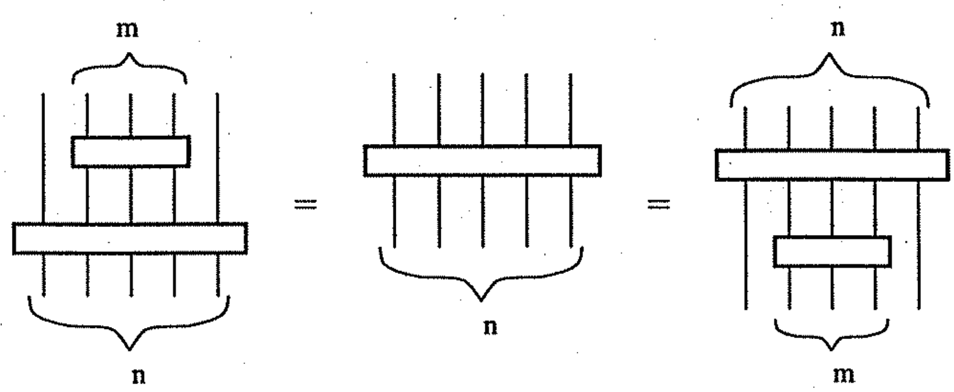
The Jones-Wenzl projector $p_n = \iota_n \circ \pi_n$ is depicted by a box with n lines entering the bottom and n lines leaving the top:



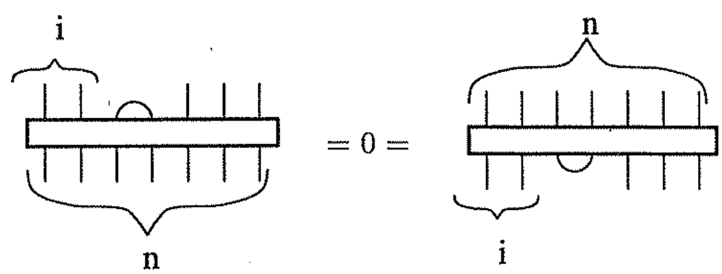
Theorem 1.4 can be easily reformulated in the graphical language, with $T(s)$ being a diagram of the positive braid representing the permutation s. Thus a

box marked n is a sum of $n!$ diagrams corresponding to all possible permutations weighted with an appropriate power of q.

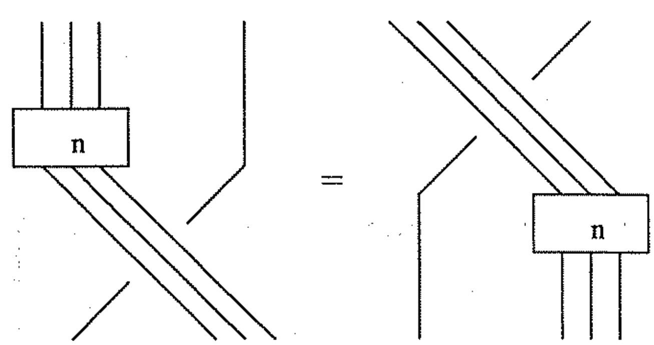
Since p_n is a projector, its iterations are all equal. More generally, Theorem 1.4 implies



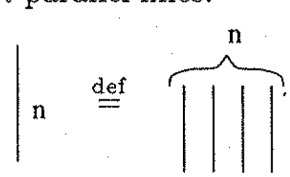
It also follows that the composition of a projector with a contraction ϵ_1 or a generation δ_1 operator yield zero:



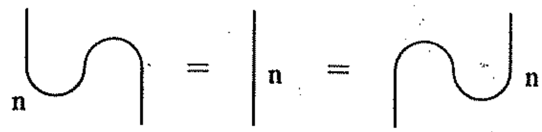
All operations of framed isotopy preserve the intertwiner associated to the diagram. For example, we can move p_n under or over a line:



A curve marked by n denotes n parallel lines:



Curves can bend arbitrarily:

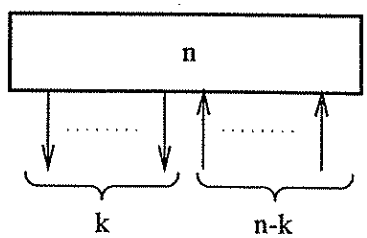


We omit writing down a full set of generating moves for the diagrammatic calculus and instead refer the reader to [KaL].

Next we consider the diagrams for the dual canonical bases in the irreducible representations $V_n, n = 0, 1, \ldots$. For $v^{n-2k} \in V_n$ one has

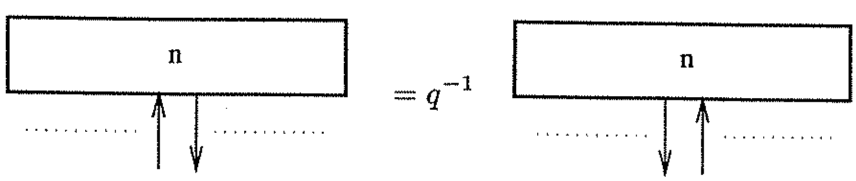
$$v^{n-2k} = \pi_n((v^{-1})^{\otimes k} \otimes (v^1)^{\otimes(n-k)})$$

Thus, we depict v^{n-2k} by



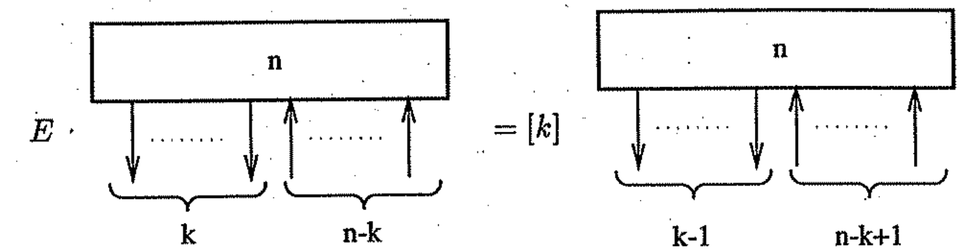
The representation V_n can be viewed as the linear span of such diagrams with $k = 0, 1, ..., n$.

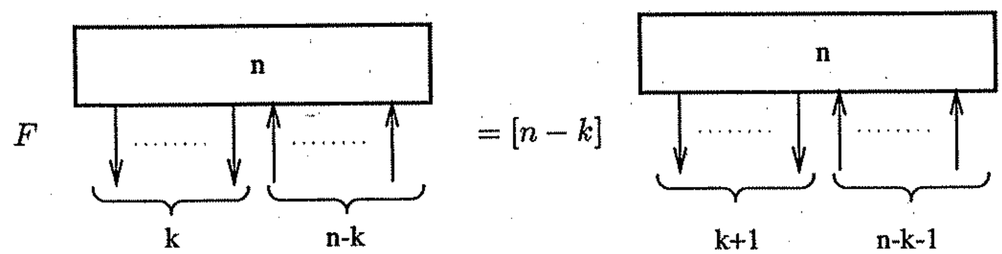
We can also associate a diagram to the element $\pi_n(v^{s_1} \otimes ... \otimes v^{s_n})$ of V_n with an arbitrary sequence $s = (s_1, ..., s_n)$ of $s_i \in \{\pm 1\}, i = 1, ..., n$, according to our rule of orientation of arrows. Then for different orders of arrows we have the following identification, implied by the formula for π_n in Proposition 1.2



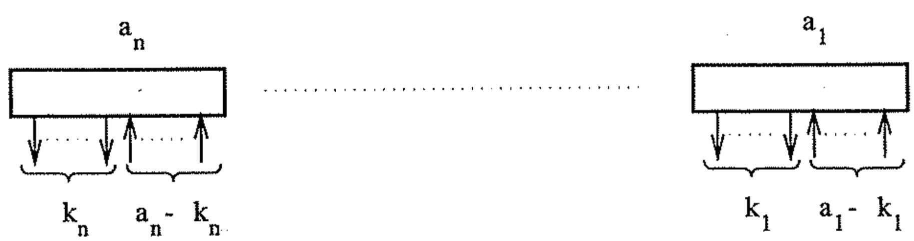
The order of arrows in the left and right pictures are the same except for the two arrows that are shown.

The action of the generators E (resp. F) of $\mathbf{U}$ can be visualized as the reversal of the rightmost (resp. leftmost) arrow pointing down (resp. up)

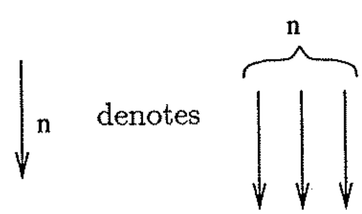




The tensor product $v^{a_1-2k_1} \otimes ... \otimes v^{a_n-2k_n}$ is depicted by placing diagrams for $v^{a_1-2k_1}, ..., v^{a_n-2k_n}$ in parallel

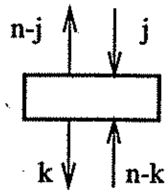


It is convenient to introduce a shorthand for n compatibly oriented parallel lines:



Next we will consider a graphical calculus of diagrams in which all the external lines are oriented. We will assume that such diagrams lie inside a horizontal strip $\mathbb{R} \times [0,1]$ in the (x,y)-plane and that the oriented lines are attached to the top and bottom of the diagrams. These diagrams can be evaluated by writing each projector as a linear combination of positive braids (Theorem 1.4) and then using the graphical counterparts (see section 2.1) of relations (1.15) to decompose the diagram into a linear combination of diagrams consisting of simple arcs. Each such diagram can be evaluated as in Section 2.1.

Formulas (1.24) imply



$$= \delta_{kj} \begin{bmatrix} n \\ k \end{bmatrix}^{-1}$$

This diagram can be viewed as the graphical realization of the scalar product $< v^{n-2k}, v^{n-2j} >= \delta_{kj} \begin{bmatrix} n \\ k \end{bmatrix}^{-1}$.

Generally, if a is a diagram depicting an intertwiner (denoted by the same letter a)

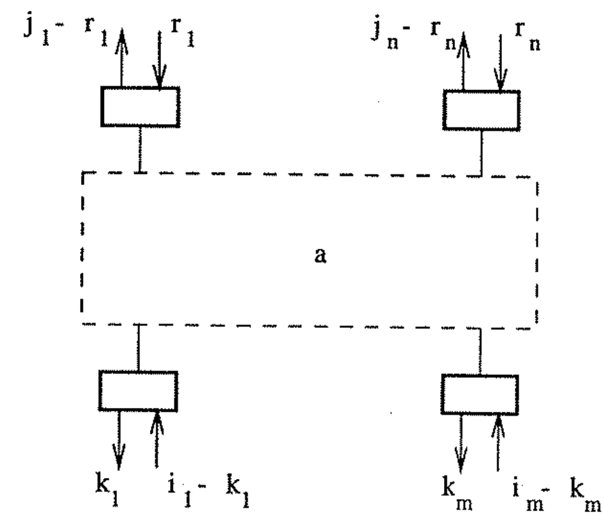
$$a : V_{i_1} \otimes ... \otimes V_{i_m} \to V_{j_1} \otimes ... \otimes V_{j_n},$$

the matrix coefficients of a admit the following geometric interpretation.

THEOREM 2.1. *The matrix coefficient*

$$< a(v^{i_1-2k_1} \otimes ... \otimes v^{i_m-2k_m}), v^{j_n-2r_n} \otimes ... \otimes v^{j_1-2r_1} >$$

is equal to the value of the following diagram

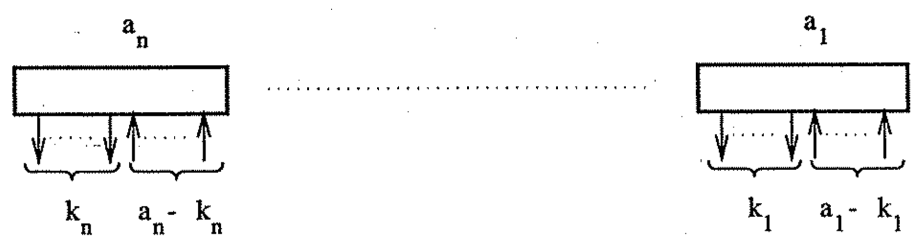


Proof: Theorem follows from the fact that $\pi_{i_l}((v^{-1})^{\otimes k_l} \otimes (v^1)^{\otimes (i_l-k_l)}) = v^{i_l-2k_l}, 1 \leq l \leq m$, and the coefficient of $(v^1)^{\otimes (j_l-r_l)} \otimes (v^{-1})^{\otimes r_l}$ in the decomposition of $\iota_n(v^{j_r-2r_l}), 1 \leq l \leq n$, is 1. □

2.3. Dual canonical basis

Remarkably, the dual canonical basis in the tensor product $V = V_{a_n} \otimes ... \otimes V_{a_1}$ and the action of the generators E and F in the dual basis admit a simple geometric interpretation, which we now describe.

Fix positive integers $a_1, ..., a_n$ and nonnegative integers $k_1, ...k_n, k_i \leq a_i, 1 \leq i \leq n$. Let us depict $v^{a_n-2k_n} \heartsuit ... \heartsuit v^{a_1-2k_1} \in B^{a_n} \heartsuit ... \heartsuit B^{a_1}$. We start with a diagram for $v^{a_n-2k_n} \otimes ... \otimes v^{a_1-2k_1}$:



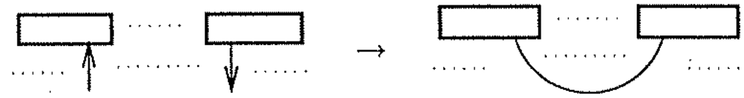
The diagram for $v^{a_n-2k_n} \heartsuit ... \heartsuit v^{a_1-2k_1}$ is constructed by closing off some of the pairs of arrows into arcs. Namely, we repeat the following procedure several times ($[\frac{a_1+...+a_n}{2}]$ times is enough):

Suppose the diagram has a pair (up arrow, down arrow) such that

(a) The up arrow is to the left of the down arrow.

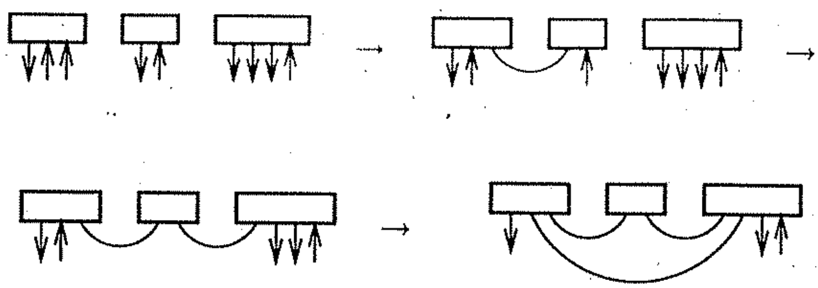
(b) No arrows lie between the two arrows.

Then we connect the two arrows into a simple unoriented arc that does not intersect anything.



Now repeat the procedure with the new diagram. Note that we allow arcs in between arrows that we connect. We stop when we fail to find a pair satisfying conditions (a)-(b). This will happen when all down arrows are to the left of all up arrows.

Example:



If at some moment there is a choice between several pairs, we choose any one of them. Clearly, the final diagram does not depend on the order in which we pick the pairs.

The final diagram defines (by considering each arc as an intertwiner δ_1) a vector in $V_{a_n} \otimes ... \otimes V_{a_1}$ By abuse of notation denote this vector by $v^{a_n-2k_n}\heartsuit...\heartsuit v^{a_1-2k_1}$. Also, we use the same notation for the diagram representing this vector. Algebraic expressions for these vectors are given by formulas in the statements of Theorems 1.8,1.9.

We will prove in Section 2.4 that this is indeed the vector $v^{a_n-2k_n}\heartsuit...\heartsuit v^{a_1-2k_1}$ of the dual canonical basis $B^{a_n}\heartsuit...\heartsuit B^{a_1}$ as defined in Chapter 1.

The diagrammatic calculus provides a geometric way to compute the action of E and F in the dual canonical basis.

First, denote by u the bijection of sets

$$u : B^{a_n}\heartsuit...\heartsuit B^{a_1} \to B^{a_1}\heartsuit...\heartsuit B^{a_n}$$

given by reflecting a diagram about a vertical axis and reversing orientations of all arrows. In the notation we suppress the dependence of u on $a_1, ..., a_n$. We have $u^2 = 1$. Denote by the same letter u the induced isomorphism of linear spaces

$$u : V_{a_n} \otimes ... \otimes V_{a_1} \to V_{a_1} \otimes ... V_{a_n}$$

Pick a diagram α of a dual canonical vector in $B_{a_n}\heartsuit...\heartsuit B_{a_1}$. We numerate down arrows of α from left to right by $1, 2, ..., \downarrow(\alpha)$ where $\downarrow(\alpha)$ is the number of down arrows of α. Similarly, $\uparrow(\alpha)$ is the number of up arrows of α. For i such that $1 \leq i < \downarrow(\alpha)$ denote by $E_{(i)}(\alpha)$ a diagram obtained from α by connecting the i-th and $i+1$-st down arrows of α into a simple arc. If $\downarrow(\alpha) > 0$, denote by $E_{(\downarrow(\alpha))}(\alpha)$ a diagram obtained by reversing the orientation of the rightmost down arrow of α. Finally, if $i \notin \{1, 2, ..., \downarrow(\alpha)\}$ set $E_{(i)}(\alpha) = 0$. Define the action of E by

$$E(\alpha) = \sum_{i \in \mathbb{Z}} [i] E_{(i)}(\alpha)$$

Define the action of $K^{\pm 1}$ by

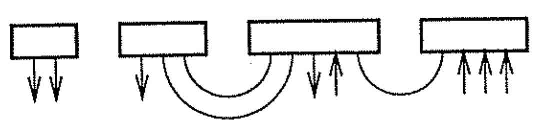
$$K^{\pm 1}(\alpha) = q^{\pm(\uparrow(\alpha) - \downarrow(\alpha))}\alpha$$

Next we introduce the action of F:

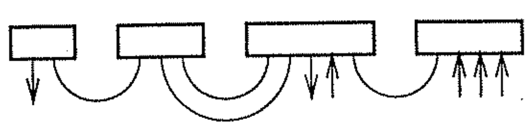
$$F(\alpha) \stackrel{\text{def}}{=} u(E(u(\alpha)))$$

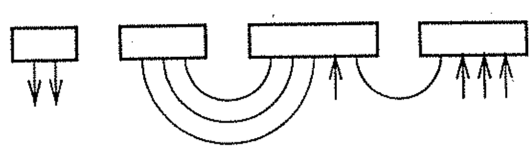
That is, take the definition of the action of E, change down to up, up to down, left to right everythere and you get the definition of the action of F.

Example: Let α_0 be



Then

$E_{(2)}(\alpha_0) =$ 

$E_{(3)}(\alpha_0) =$ 

$E_{(4)}(\alpha_0) =$ 

Also, $E_{(1)}(\alpha_0) = 0$ because the diagram for $E_{(1)}(\alpha_0)$ contains an arc with both ends attached to the bottom of the same projector. Thus

$$E(\alpha_0) = [2]E_{(2)}(\alpha_0) + [3]E_{(3)}(\alpha_0) + [4]E_{(4)}(\alpha_0)$$

F acts on α_0 as follows: $F(\alpha_0) = [3]F_{(3)}(\alpha_0) + [4]F_{(4)}(\alpha_0)$ where

$F_{(3)}(\alpha_0) =$ 

$F_{(4)}(\alpha_0) =$ 

THEOREM 2.2. *The above definition of the action of* E, F *and* $K^{\pm 1}$ *yields a representation of* $\mathbf{U}$ *on the vector space* $V_{a_n} \otimes \dots \otimes V_{a_1}$.

Proof. We can verify that the operators $E, F, K^{\pm 1}$ satisfy the quantum group relations (1.1). All relations except for

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

are obvious and we only check this one.

We have

$$EF(\alpha) = \sum_x \sum_y [x][y]E_{(x)}F_{(y)}(\alpha)$$

and

$$FE(\alpha) = \sum_y \sum_x [x][y]F_{(y)}E_{(x)}(\alpha)$$

In both sums, only finitely many terms are non-zero.

Drawing pictures, it is easy to verify that $E_{(x)}F_{(y)}(\alpha) = F_{(y)}E_{(x)}(\alpha)$ unless $x = \downarrow(\alpha) + 1$ and $y = \uparrow(\alpha)$. In the latter case

$$E_{(\downarrow(\alpha)+1)}F_{(\uparrow(\alpha))}(\alpha) = \begin{cases} 0 \text{ if } \uparrow(\alpha) = 0 \\ [\uparrow(\alpha)][\downarrow(\alpha) + 1]\alpha \text{ otherwise} \end{cases}$$

and

$$F_{(\uparrow(\alpha)+1)}E_{(\downarrow(\alpha))}(\alpha) = \begin{cases} 0 \text{ if } \downarrow(\alpha) = 0 \\ [\downarrow(\alpha)][\uparrow(\alpha)+1]\alpha \text{ otherwise} \end{cases}$$

Therefore,

$$EF(\alpha) - FE(\alpha) = E_{(\downarrow(\alpha)+1)}F_{(\uparrow(\alpha))}(\alpha) - F_{(\uparrow(\alpha)+1)}E_{(\downarrow(\alpha))}(\alpha) =$$
$$[\uparrow(\alpha) - \downarrow(\alpha)](\alpha) = \frac{K-K^{-1}}{q-q^{-1}}(\alpha)$$

□

Restricting the action of E and F described above to the case $r=2$ we obtain proposition 1.6(iii).

PROPOSITION 2.3. *This action coincides with the standard action of E and F in the representation $V_{a_n} \otimes ... \otimes V_{a_1}$.*

Proof. First , it is enough to prove these formulas in the case of the tensor product $V_1^{\otimes(a_1+...+a_n)}$ and then use the projection $\pi_{a_n} \otimes ... \otimes \pi_{a_1}$ to deduce it for an arbitrary tensor product. Next, it suffices to check the formulas for the action of E and F in the special case when the diagram representing an element of the dual canonical basis of $V_1^{\otimes m}$ has no closed arcs, and this is a straightforward computation. □

2.4. Proof of the graphical presentation of the dual canonical basis

Given an intertwiner $T : V_1^{\otimes m} \to V_1^{\otimes n}$ (or a diagram that represents T) and two sequences $(t_1, ..., t_m), (s_1, ..., s_n)$ of 1's and -1's, the matrix coefficient

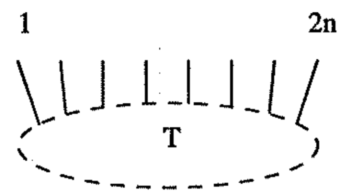
$$< T(v_{t_1} \otimes ... \otimes v_{t_m}), v_{s_1} \otimes ... \otimes v_{s_n} >$$

is called *the evaluation* of (T, t, s) and denoted $ev(T, t, s)$.

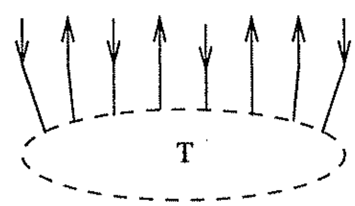
It turns out that, for a diagram T and sequences t and s of 1's and -1's, the evaluation $ev(T, t, s)$ can be computed as a value of a diagram without external arcs, that is, a diagram that defines an intertwiner from the trivial representation

V_0 to V_0. The idea is to add a projector and plug the external arcs of T into the projector in a special way. Below we restrict to the case $m = 0$. The case of general m can be easily reduced to this particular case.

Let $s = (s_1, ..., s_{2n})$ be a sequence of 1's and -1's such that $s_1 + ... + s_{2n} = 0$. Let T be an arbitrary element of $\mathrm{Inv}_{\mathbf{U}}(V_1^{\otimes(2n)})$. For convenience we suppose that T is given by some diagram



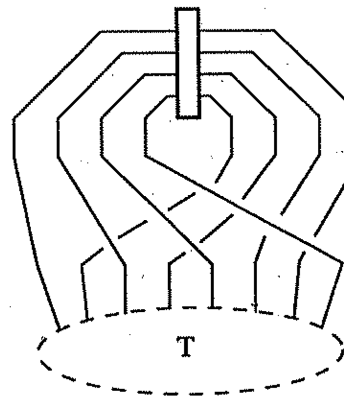
The sequence s and the diagram T define a diagram obtained by orienting $2n$ ends of T up or down according to s.



Denote this diagram by (T, s). This diagram defines a matrix coefficient of the intertwiner T. As before, call this coefficient *the evaluation* of (T, s) and denote by $ev(T, s)$.

A diagram is called *closed,* or a diagram without external arcs if the associated element goes from the one-dimensional representation V_0 to V_0. Thus, a closed diagram defines a number that we call *the evaluation* of the diagram and also denote by ev.

Define the closure of (T, s) as a diagram obtained from (T, s) by adding a projector of size n, attaching all down arrows of (T, s) in parallel to the left side of the projector and all up arrows to the right side of the projector, so that the down arrows go over the up arrows, and finally erasing the orientations:



Denote the closure of (T, s) by $cl(T, s)$.

Remark. This "closure" construction is a q-version of the one used by J.P.Missouris to evaluate Clebsch-Gordan coefficients for $\mathfrak{sl}_2$ (see [M]).

The closure $cl(T, s)$ is a closed diagram as defined above.

THEOREM 2.4. *For any T and s as above*

$$\frac{ev(cl(T,s))}{ev(T,s)} = (-1)^n q^{-\frac{\|s\|_-}{2}}[n+1] \tag{2.1}$$

Recall that $\|s\|_-$ is the number of pairs (i, j), $\quad 1 \le i < j \le 2n, s_i = -1, s_j = 1$.

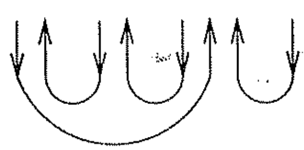
Proof. By linearity it is enough to prove this theorem as T runs over basis vectors of $\mathrm{Inv}_{\mathbf{U}}(V_1^{\otimes(2n)})$ for some basis. Our choice is the dual canonical basis. Pick α from the dual canonical basis of $\mathrm{Inv}_{\mathbf{U}}(V_1^{\otimes(2n)})$. The diagram of α consists of n simple, pairwise nonintersecting arcs. Consider the diagram (α, s). Denote the number of arcs in (α, s) oriented clockwise (respectively, anticlockwise) by $y(\alpha, s)$ (respectively, by $x(\alpha, s)$). Then

$$x(\alpha, s) + y(\alpha, s) = n$$

Example. Let $s = (-1, 1, -1, 1, -1, 1, 1, -1)$,

$\alpha_1 =$ 

Then

$(\alpha_1, s) =$ 

and $x(\alpha_1, s) = 1, y(\alpha_1, s) = 3$.

Returning to α and s, we have

$$ev(\alpha, s) = (-q)^{-x(\alpha,s)}$$

It is also easy to verify that

$$ev(cl(\alpha, s)) = (-1)^{x(\alpha,s)+n} q^{-x(\alpha,s)-\frac{\|s\|-}{2}} [n+1]$$

Here $(-1)^n[n+1]$ comes from evaluating the closure of the Jones-Wenzl projector p_n (see [KaL], for instance). $\square$

Therefore, the diagrams $cl(T, s)$ and (T, s) compute essentially the same number, up to a simple constant that depends only on s and not on $T \in \mathrm{Inv}_{\mathbf{U}}(V_1^{\otimes(2n)})$.

Remark. The representation-theoretical meaning of the closure construction is that, normalized according to (2.1), it provides the **U**-module projection $V_1^{\otimes 2n} \to \mathrm{Inv}_{\mathbf{U}}(V_1^{\otimes 2n})$.

We are ready now to prove that the set of vectors given by the diagrams described in Section 2.3 is indeed the dual canonical basis. The proof consists of verifying condition (1.36) of Theorem 1.7(a). Condition (1.37) obviously holds.

We are given a tensor product $V_{a_1} \otimes ... \otimes V_{a_n}$. This tensor product has an elementary basis

$$\{v^{l_1} \otimes ... \otimes v^{l_n}\}, -a_i \leq l_i \leq a_i, l_i = a_i (\bmod 2)$$

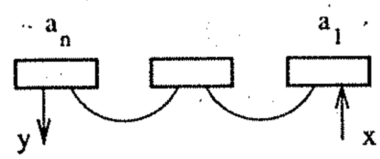
Fix $k_1, ..., k_n$ such that

$$-a_i \leq k_i \leq a_i, \quad k_i = a_i (\bmod 2)$$

Consider the dual tensor product $V_{a_n} \otimes ... \otimes V_{a_1}$. We want to show

$$\overline{\Psi}^{(n)} v^{k_n} \heartsuit ... \heartsuit v^{k_1} = v^{k_n} \heartsuit ... \heartsuit v^{k_1}$$

where $v^{k_n}\heartsuit...\heartsuit v^{k_1} \in V_{a_n} \otimes ... \otimes V_{a_1}$ is given by a diagram as described in Section 2.3. We depict schematically this diagram by



with x (resp. y) being the number of up (resp. down) arrows.

Formula (1.36) is equivalent to

$$< v^{l_1} \otimes ... \otimes v^{l_n}, \overline{\Psi}^{(n)} v^{k_n}\heartsuit...\heartsuit v^{k_1} >=< v^{l_1} \otimes ... \otimes v^{l_n}, v^{k_n}\heartsuit...\heartsuit v^{k_1} >$$

for all $v^{l_1} \otimes ... \otimes v^{l_n}$ from the elementary basis of $V_{a_1} \otimes ... \otimes V_{a_n}$. We have

$$\overline{\Psi}^{(n)} = \overline{\Theta}^{(n)} \circ \sigma^{\otimes n} \quad \text{and} \quad \mathcal{R}^{(n)} = C^{(n)}\Theta^{(n)}$$

Thus we want to show

$$< v^{l_1} \otimes ... \otimes v^{l_n}, \overline{\Theta}^{(n)} \sigma^{\otimes n}(v^{k_n}\heartsuit...\heartsuit v^{k_1}) >=< v^{l_1} \otimes ... \otimes v^{l_n}, v^{k_n}\heartsuit...\heartsuit v^{k_1} >$$

or

$$< v^{l_1} \otimes ... \otimes v^{l_n}, \overline{C^{-1}}^{(n)} \overline{\mathcal{R}}^{(n)} \sigma^{\otimes n}(v^{k_n}\heartsuit...\heartsuit v^{k_1}) >=< v^{l_1} \otimes ... \otimes v^{l_n}, v^{k_n}\heartsuit...\heartsuit v^{k_1} >$$

This is equivalent to (using that $\overline{C^{-1}}^{(n)} = C^{(n)}$)

(2.2) $$< \overline{\mathcal{R}}^{(n)} C^{(n)} v^{l_1} \otimes ... \otimes v^{l_n}, \sigma^{\otimes n}(v^{k_n}\heartsuit...\heartsuit v^{k_1}) >=< v^{l_1} \otimes ... \otimes v^{l_n}, v^{k_n}\heartsuit...\heartsuit v^{k_1} >$$

We have

$$C^{(n)} v^{l_1} \otimes ... \otimes v^{l_n} = \left(\prod_{i<j} q^{-\frac{l_i l_j}{2}}\right) v^{l_1} \otimes ... \otimes v^{l_n}$$

Hence, (2.2) is equivalent to

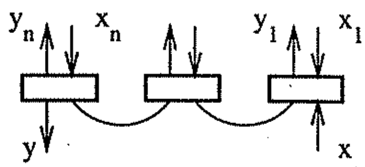
$$< \mathcal{R}^{(n)} v^{l_1} \otimes ... \otimes v^{l_n}, v^{k_n}\heartsuit...\heartsuit v^{k_1} >= q^{-\frac{1}{2}\sum_{i<j} l_i l_j} \overline{< v^{l_1} \otimes ... \otimes v^{l_n}, v^{k_n}\heartsuit...\heartsuit v^{k_1} >}$$

Let

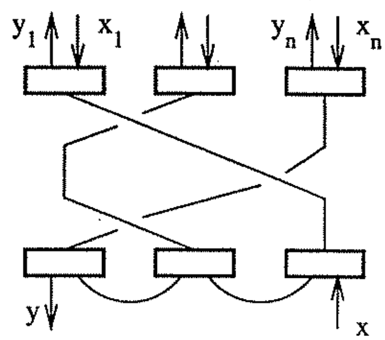
$$x_i = \frac{a_i - l_i}{2}, \quad y_i = \frac{a_i + l_i}{2}$$

Introduce the following two diagrams (S and T):

S



T

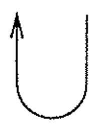


From what was said earlier we deduce

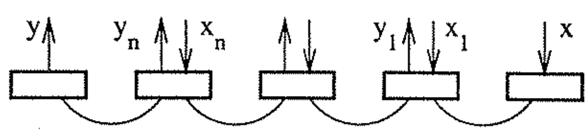
$$\text{(2.3)} \qquad \begin{aligned} ev(S) &=< v^{l_1} \otimes \ldots \otimes v^{l_n}, v^{k_n} \heartsuit \ldots \heartsuit v^{k_1} > \\ ev(T) &=< \mathcal{R}^{(n)} v^{l_1} \otimes \ldots \otimes v^{l_n}, v^{k_n} \heartsuit \ldots \heartsuit v^{k_1} > \end{aligned}$$

Let us now relate $ev(S)$ and $ev(T)$.

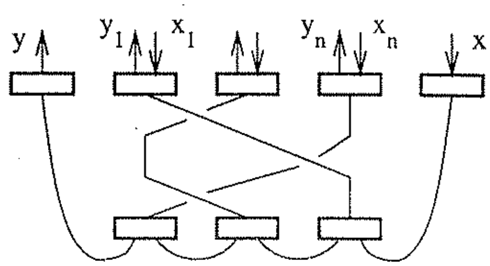
First, because

 $= 1,$

evaluations of S and T are equal to evaluations of the following diagrams



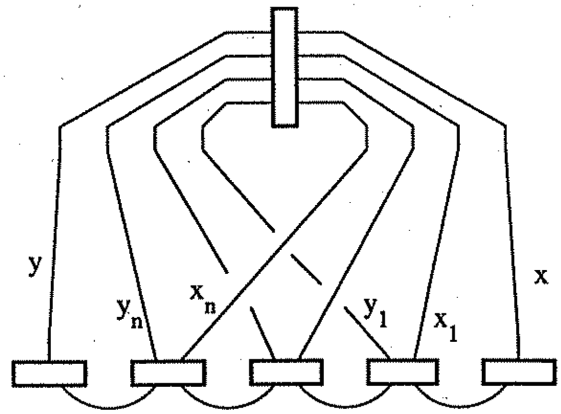
for S,



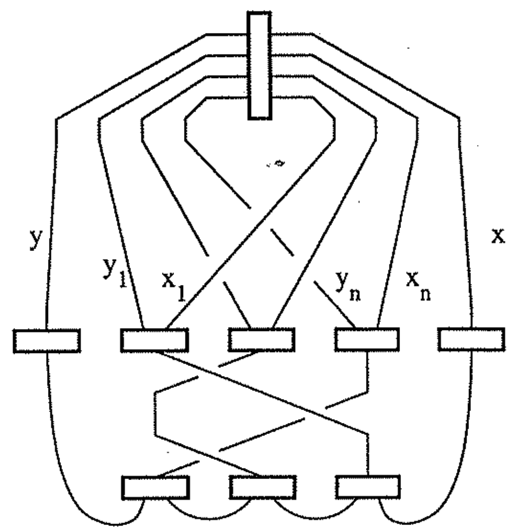
for T.

Let X and Y be the closures of these diagrams:

X



Y



Then by Theorem 2.4

$$ev(X) = (-1)^k[k+1]q^{-\frac{1}{2}\sum_{i>j} x_i y_j} ev(S)$$

$$ev(Y) = (-1)^k[k+1]q^{-\frac{1}{2}\sum_{i<j} x_i y_j} ev(T)$$

Here

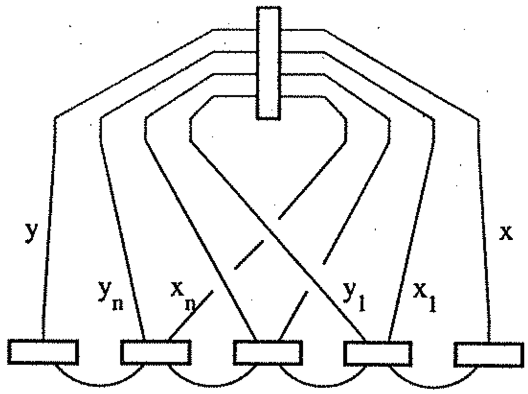
$$k = x_1 + \ldots + x_n + x = y_1 + \ldots + y_n + y = \frac{a_1 + \ldots + a_n}{2}$$

Untwisting the central part of the diagram Y we obtain

$$ev(Y) = q^{-\frac{1}{2}\sum_{i<j}(x_i x_j + y_i y_j)} ev(X')$$

where

X'



The diagram X' is constructed from X by changing all overcrossings into under-crossings and vice versa. Therefore,

$$ev(X') = \overline{ev(X)}$$

where overline denotes the involution on $\mathbb{C}(q)$ that sends q to q^{-1}.

Solving the above equations for $ev(T)$ and $ev(S)$ we obtain

$$ev(T) = \overline{ev(S)} q^{-\frac{1}{2}\sum_{i<j}(y_i - x_i)(y_j - x_j)}.$$

Note that $l_i = y_i - x_i$. Hence,

$$ev(T) = \overline{ev(S)} q^{-\frac{1}{2}\sum_{i<j} l_i l_j}$$

Substituting scalar products (2.3) for $ev(T)$ and $ev(S)$ we get

$$< \mathcal{R}^{(n)} v^{l_1} \otimes \ldots \otimes v^{l_n}, v^{k_n} \heartsuit \ldots \heartsuit v^{k_1} >= q^{-\frac{1}{2}\sum_{i<j} l_i l_j} \overline{< v^{l_1} \otimes \ldots \otimes v^{l_n}, v^{k_n} \heartsuit \ldots \heartsuit v^{k_1} >}$$

We verified that our basis satisfies (1.36). This completes the proof of Theorems 1.8 and 1.9. Thus, the diagrammatical basis described in Section 2.3 is indeed dual canonical. □

The formula (1.36) can also be verified without using the closure construction and working directly with non-closed diagrams. The second approach better explains why in the diagram for a dual canonical basis vector all down arrows are to the left of all up arrows. The reason is that the braiding matrix acts on $v_{-1} \otimes v_1$ in a particularly simple way: $\check{R}_{11}(v_{-1} \otimes v_1) = q^{\frac{1}{2}} v_1 \otimes v_{-1}$.

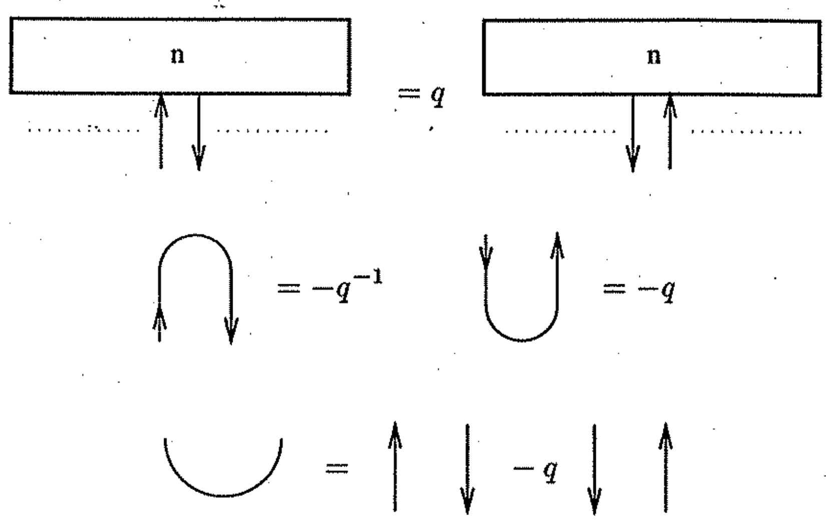
CHAPTER III

GRAPHICAL CALCULUS AND CANONICAL BASIS

3.1. Conjugated graphical calculus

In this chapter we will write explicit expressions for canonical bases in tensor products of $U_q(\mathfrak{sl}_2)$ modules and in the invariants of these products. Throughout the chapter we will use comultiplication Δ rather than $\overline{\Delta}$ to define the $U_q(\mathfrak{sl}_2)$ module structure on tensor products of $U_q(\mathfrak{sl}_2)$ modules. From Chapter II we already have explicit formulas for dual canonical basis vectors and to verify that the vectors given by formulas of theorems 1.12 and 1.13 are canonical we simply compute scalar products of each of these vectors with dual canonical vectors and check that the products vanish except for one case when the product is 1.

We again use grahical calculus as a tool, by interpreting the scalar product graphically and then applying the rules of graphical calculus to simplify the diagram. Because we work with the canonical basis and, consequently, with the tensor product structure given by the comultiplication Δ, we need to make a minor modification of the graphical calculus. In fact, all we do is change q to q^{-1} everywhere. Namely, change q to q^{-1} in formulas (1.14) and (1.24). That will modify the diagrammatic formulas of Chapter II as follows



Remark The braiding is not used in this chapter at all, so we do not concern ourselves with describing the way it changes under the conjugation q to q^{-1}.

Recall that we have a bilinear pairing

$$< y, x > \in \mathbb{C}(q)$$

for $y \in V_{a_1} \otimes ... \otimes V_{a_n}$ and $x \in V_{a_n} \otimes ... \otimes V_{a_1}$ where $<,>$ is just the direct product of bilinear pairings $V_i \otimes V_i \to \mathbb{C}(q)$ defined in Section 1.2.4.

To prove theorem 1.12 we will need a graphical presentation of the linear form

$$\psi_x(y) = < y, x >, \quad y \in V_{a_1} \otimes ... \otimes V_{a_n}$$

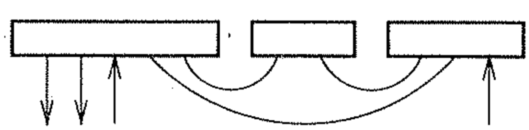
$$\psi_x \quad : \quad V_{a_1} \otimes ... \otimes V_{a_n} \to \mathbb{C}(q)$$

where x is an element of the dual canonical basis of $V_{a_n} \otimes ... \otimes V_{a_1}$. Let x be a dual canonical basis vector,

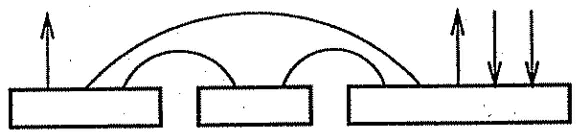
$$x = v^{a_n - 2m_n} \heartsuit ... \heartsuit v^{a_1 - 2m_1}$$

for some $m_1, ..., m_n, 0 \leq m_i \leq a_i$. Form the diagram for x as described in Section 2.3. An example is depicted below.

x



Now take the diagram of x, rotate it by 180 degrees and change the orientation of all oriented arrows. It is easy to verify that we obtain a diagram for the linear form ψ_x.

ψ_x 

In the next section we will use this presentation of ψ_x for a dual canonical basis vector x to check that the rules of theorem 1.12 give us formulas for the canonical basis.

3.2. Jones-Wenzl projectors and canonical bases in tensor products

Recall from Section 1.4 that we denoted by $v(x_1, y_1; ...; x_k, y_k)$ the element of the canonical basis of $V_1^{\otimes n}, n = \sum_{i=1}^k x_i + y_i$ with the lexicographically highest term $v_1^{\otimes x_1} \otimes v_{-1}^{\otimes y_1} \otimes ... \otimes v_1^{\otimes x_k} \otimes v_{-1}^{\otimes y_k}$. We allow some of x_i and y_i to be 0. Thus,

$$v(x_1, y_1; ...; x_k, y_k) = v_1^{\otimes x_1} \Diamond v_{-1}^{\otimes y_1} \Diamond ... \Diamond v_1^{\otimes x_k} \Diamond v_{-1}^{\otimes y_k}. \tag{3.1}$$

We first recall the following lemma from Section 1.4.2.

LEMMA 1.11.

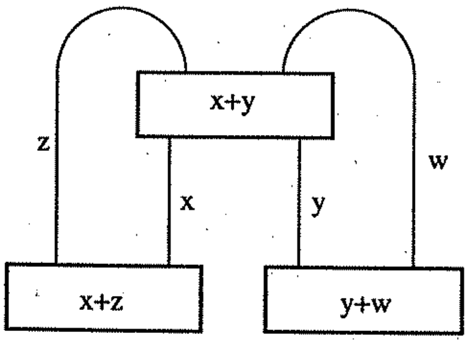
$(i)\quad v(0, y_1; x_2, ...; x_k, y_k) = v_{-1}^{\otimes y_1} \otimes v(x_2, y_2, ..., x_k, y_k),$

$(ii)\quad v(x_1, y_1; ..., y_{k-1}; x_k, 0) = v(x_1, y_1, ..., x_{k-1}, y_{k-1}) \otimes v_1^{x_k}.$

□

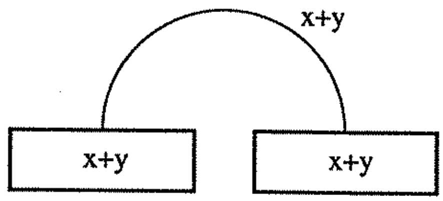
We will also need the following

LEMMA 3.1. *Let x, y, z, w be nonnegative integers and $x+z=y+w$. Then*



$=$

$=\begin{cases} \begin{bmatrix} x+y \\ x \end{bmatrix}^{-1} \end{cases}$



if $x=w, y=z$

0 *otherwise*

Lemma 3.1 is an immediate corollary of lemma 4 in [MV].

Recall that $1^{\otimes m}$ denotes the identity operator $V_1^{\otimes m} \to V_1^{\otimes m}$. In this section we are going to prove theorem 1.12. First let us state it again.

THEOREM 1.12. *(i) If $y_{i-1} \geq x_i$ and $y_i \leq x_{i+1}$ then*

$$v(x_1, y_1; ...; x_k, y_k) = \begin{bmatrix} x_i + y_i \\ x_i \end{bmatrix} (1^{\otimes l} \otimes p_{x_i+y_i} \otimes 1^{\otimes j})$$
$$v(x_1, y_1; ...; x_{i-1}, y_{i-1} + y_i; x_i + x_{i+1}, y_{i+1}; ...; x_k, y_k)$$

where $l = \sum_{t<i} x_t + y_t, j = \sum_{t>i} x_t + y_t$.

(ii) If $y_1 \leq x_2$ then

$$v(x_1, y_1; ...; x_k, y_k) =$$
$$= \begin{bmatrix} x_1 + y_1 \\ x_1 \end{bmatrix} (p_{x_1+y_1} \otimes 1^{n-x_1-y_1})(v_{-1}^{\otimes y_1} \otimes v(x_1 + x_2, y_2; ...; x_k, y_k)),$$

(iii) If $y_{k-1} \geq x_k$ then

$$v(x_1, y_1; ...; x_k, y_k) =$$
$$= \begin{bmatrix} x_k + y_k \\ x_k \end{bmatrix} (1^{\otimes n-x_k-y_k} \otimes p_{x_k+y_k})(v(x_1, y_1; ...; x_{k-1}, y_{k-1} + y_k) \otimes v_1^{\otimes x_k}).$$

Before engaging upon the proof, we would like to show the reader what the canonical basis vector $v(x_1, y_1; ...; x_k, y_k)$ looks like for small values of k.

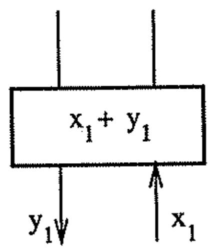
Example: Vectors $v(x_1, y_1; \ldots; x_k, y_k)$ for $k \leq 3$.

(i) $k = 1$. In this case

$$v(x_1, y_1) = \begin{bmatrix} x_1 + y_1 \\ x_1 \end{bmatrix} p_{x_1+y_1}(v_{-1}^{\otimes y_1} \otimes v_1^{\otimes x_1})$$

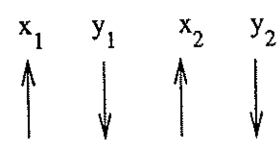
or, graphically,

$$v(x_1, y_1) = \begin{bmatrix} x_1 + y_1 \\ x_1 \end{bmatrix}$$



(ii) $k = 2$. The tensor product vector

$v_1^{\otimes x_1} \otimes v_{-1}^{\otimes y_1} \otimes v_1^{\otimes x_2} \otimes v_{-1}^{\otimes y_2}$ is depicted by



and the canonical basis vectors are given by

$v(x_1, y_1; x_2, y_2) =$

$= C_1(p_{x_1+y_1} \otimes 1^{\otimes(x_2+y_2)})(1^{\otimes y_1} \otimes p_{x_1+x_2+y_2})(v_{-1}^{\otimes(y_1+y_2)} \otimes v_1^{\otimes(x_1+x_2)})$

if $y_1 \leq x_2$,

$v(x_1, y_1; x_2, y_2) =$

$= C_2(1^{\otimes(x_1+y_1)} \otimes p_{x_2+y_2})(p_{x_1+y_1+y_2} \otimes 1^{\otimes x_2})(v_{-1}^{\otimes(y_1+y_2)} \otimes v_1^{\otimes(x_1+x_2)})$

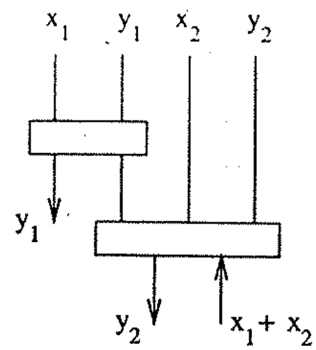
if $y_1 \geq x_2$,

where

$$C_1 = \begin{bmatrix} x_1 + y_1 \\ x_1 \end{bmatrix} \begin{bmatrix} x_1 + x_2 + y_2 \\ y_2 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} x_2 + y_2 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 + y_1 + y_2 \\ x_1 \end{bmatrix}$$

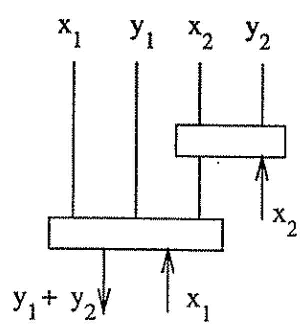
Graphically,

$$v(x_1, y_1; x_2, y_2) = C_1$$



if $y_1 \leq x_2$

$$v(x_1, y_1; x_2, y_2) = C_2$$

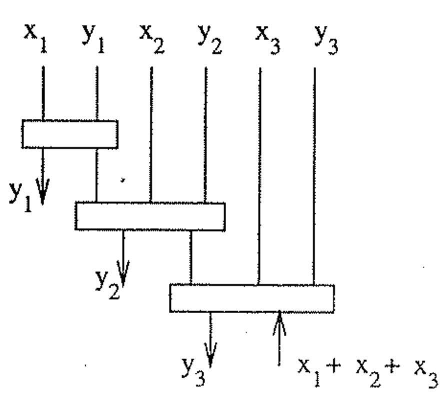


if $y_1 \geq x_2$

(iii) $k = 3$. We have 5 types of diagrams depending on the values of y_1, x_2, y_2, x_3:

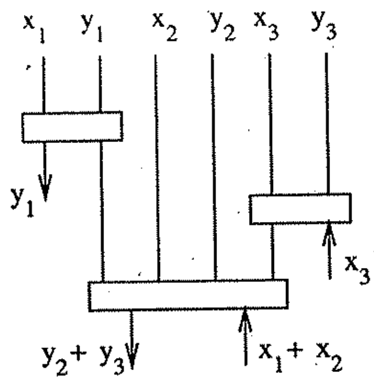
$$v(x_1, y_1; x_2, y_2; x_3, y_3) =$$

C_3



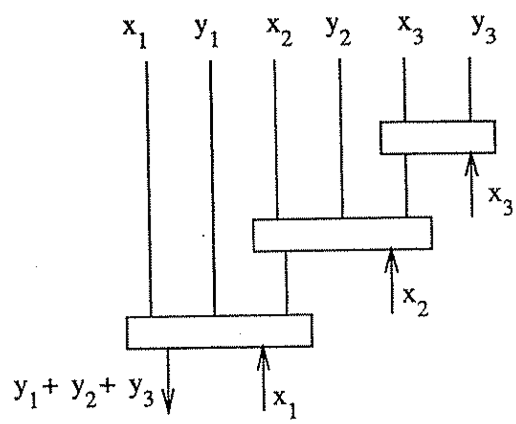
if $x_3 \geq y_2, x_2 \geq y_1$

C_4



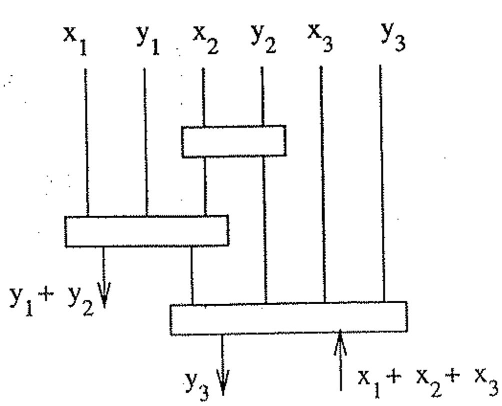
if $x_2 \geq y_1, y_2 \geq x_3$

C_5

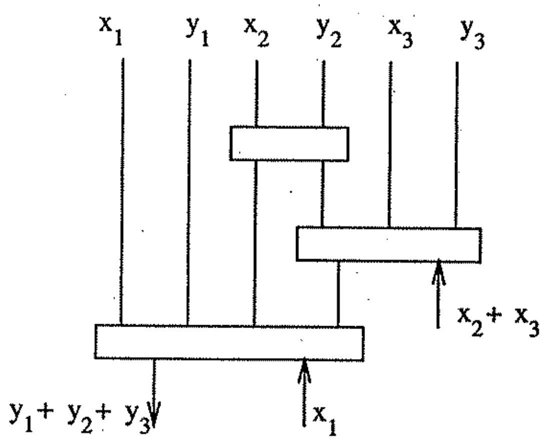


if $y_2 \geq x_3, y_1 \geq x_2$

C_6



if $y_1 \geq x_2, x_3 \geq y_2, x_2 + x_3 \geq y_1 + y_2$

C_7 

if $y_1 \geq x_2, x_3 \geq y_2, y_1+y_2 \geq x_2+x_3$,

where

$$C_3 = \begin{bmatrix} x_1+y_1 \\ y_1 \end{bmatrix}\begin{bmatrix} x_1+x_2+y_2 \\ y_2 \end{bmatrix}\begin{bmatrix} x_1+x_2+x_3+y_3 \\ y_3 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} x_1+y_1 \\ y_1 \end{bmatrix}\begin{bmatrix} x_3+y_3 \\ x_3 \end{bmatrix}\begin{bmatrix} x_1+x_2+y_2+y_3 \\ x_1+x_2 \end{bmatrix}$$

$$C_5 = \begin{bmatrix} x_3+y_3 \\ x_3 \end{bmatrix}\begin{bmatrix} x_2+y_2+y_3 \\ x_2 \end{bmatrix}\begin{bmatrix} x_1+y_1+y_2+y_3 \\ x_1 \end{bmatrix}$$

$$C_6 = \begin{bmatrix} x_2+y_2 \\ x_2 \end{bmatrix}\begin{bmatrix} x_1+y_1+y_2 \\ x_1 \end{bmatrix}\begin{bmatrix} x_1+x_2+x_3+y_3 \\ y_3 \end{bmatrix}$$

$$C_7 = \begin{bmatrix} x_2+y_2 \\ x_2 \end{bmatrix}\begin{bmatrix} x_2+x_3+y_3 \\ y_3 \end{bmatrix}\begin{bmatrix} x_1+y_1+y_2+y_3 \\ x_1 \end{bmatrix}$$

Remark: In general, the number of different types of diagrams for $v(x_1, y_1; \ldots; x_k, y_k)$ is equal to the dimension of the Temperley-Lieb algebra TL_k.

Proof of theorem 1.12.

The second and third claims of the theorem can be easily deduced from the first. Let us deduce the second claim, for example. We have a sequence $x_1, \ldots, y_k$ with

$y_1 \leq x_2$. Let $z \geq x_1$. Then

$$v_{-1}^{\otimes z} \otimes v(x_1, y_1; ...; x_k, y_k) = (\text{ by lemma 1.11 })$$

$$v(0, z; x_1, y_1; ...; x_k, y_k) = (\text{ by part } (i) \text{ of Theorem 1.12})$$

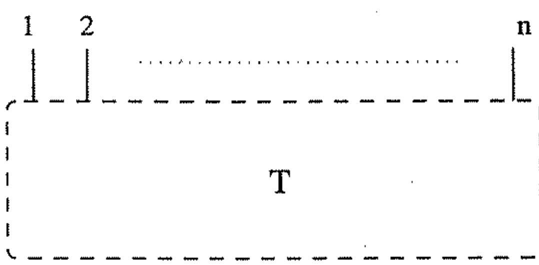
$$= \begin{bmatrix} x_1 + y_1 \\ x_1 \end{bmatrix} (1^{\otimes z} \otimes p_{x_1+y_1} \otimes 1^{\otimes(n-x_1-y_1)}) v(0, z + y_1; x_1 + x_2, y_2; ...; x_k, y_k) =$$

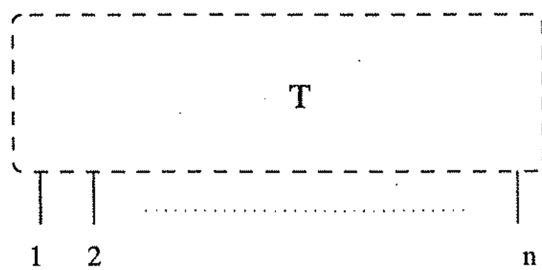
$$= \begin{bmatrix} x_1 + y_1 \\ x_1 \end{bmatrix} (1^{\otimes z} \otimes p_{x_1+y_1} \otimes 1^{\otimes(n-x_1-y_1)}) (v_{-1}^{\otimes(z+y_1)} \otimes v(x_1 + x_2, y_2; ...; x_k, y_k) =$$

$$= \begin{bmatrix} x_1 + y_1 \\ x_1 \end{bmatrix} v_{-1}^{\otimes z} \otimes (p_{x_1+y_1} \otimes 1^{\otimes(n-x_1-y_1)}) (v_{-1}^{\otimes y_1} \otimes v(x_1 + x_2, y_2; ...; x_k, y_k))$$

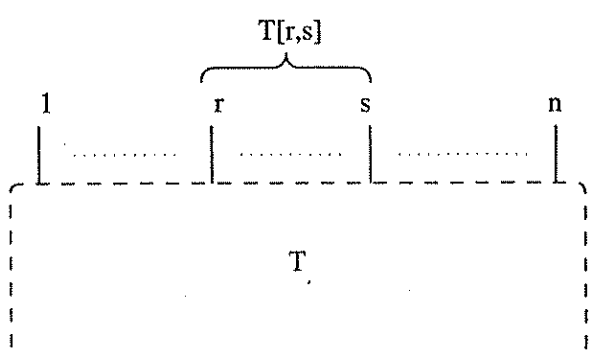
That implies (ii). Part (iii) is handled similarly.

Now a piece of notation. If a vector (respectively a covector) in $V_1^{\otimes n}$ is given by a diagram (say, a diagram T), we enumerate the top (respectively the bottom) lines of the diagram from 1 to n starting with the leftmost one.





Also, the set of lines numbered from r to s is denoted $T[r, s]$:



We will now prove part (i) of Theorem 1.12. Simultaneously with proving (i) we will prove

LEMMA 3.2.

$$(p_{x_1} \otimes p_{y_1} \otimes ... \otimes p_{x_k} \otimes p_{y_k})v(x_1, y_1; ...; x_k, y_k) = v(x_1, y_1; ...; x_k, y_k)$$

Part (i) of theorem 1.12 and lemma 3.2 are proved by simultaneous induction on the number of inversions in $v(x_1, y_1; ...; x_k, y_k)$ where the number of inversions is the number of times v_1 appears to the left of v_{-1} in the tensor product $v_1^{\otimes x_1} \otimes v_{-1}^{\otimes y_1} \otimes ... \otimes v_{-1}^{\otimes y_k}$. Thus, the number of inversions is $\sum_{i \leq j} x_i y_j$.

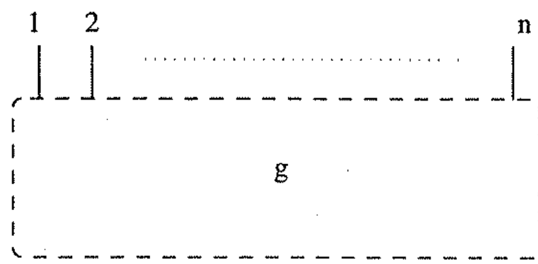
The induction base for part (i): the number of inversions is 0, all -1's are to the left of all 1's and there is nothing to prove. The induction base for lemma 3.2 is immediate as then the canonical basis vector is equal to the tensor product vector:

$$v(0, y_1; x_2, 0) = v_{-1}^{\otimes y_1} \otimes v_1^{\otimes x_2}$$

and

$$(p_{y_1} \otimes p_{x_2})(v_{-1}^{\otimes y_1} \otimes v_1^{\otimes x_2}) = v_{-1}^{\otimes y_1} \otimes v_1^{\otimes x_2}$$

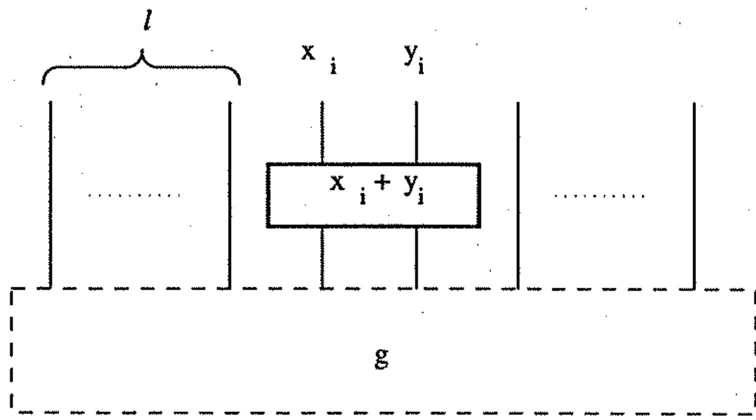
The induction step is done simultaneously for part (i) of Theorem 1.12 and for the lemma. Given a vector $v(x_1, y_1; ...; x_k, y_k)$, suppose (i) and Lemma 3.2 hold for any canonical basis vector with the inversion number less than $\sum_{i \leq j} x_i y_j$. We are also given a number i such that $y_{i-1} \geq x_i$ and $y_i \leq x_{i+1}$. We depict the vector $v(x_1, y_1; ...; x_{i-1}, y_{i-1} + y_i; x_i + x_{i+1}, y_{i+1}; ...; x_k, y_k)$, denoted for simplicity by g, by a box with dashed boundary



Then the vector

$$v = (1^{\otimes l} \otimes p_{x_i + y_i} \otimes 1^{\otimes j})g$$

where $l = x_1 + y_1 + ... + x_{i-1} + y_{i-1}$ and $j = x_{i+1} + y_{i+1} + ... + x_k + y_k$, is depicted by adding a projector to the diagram of g:



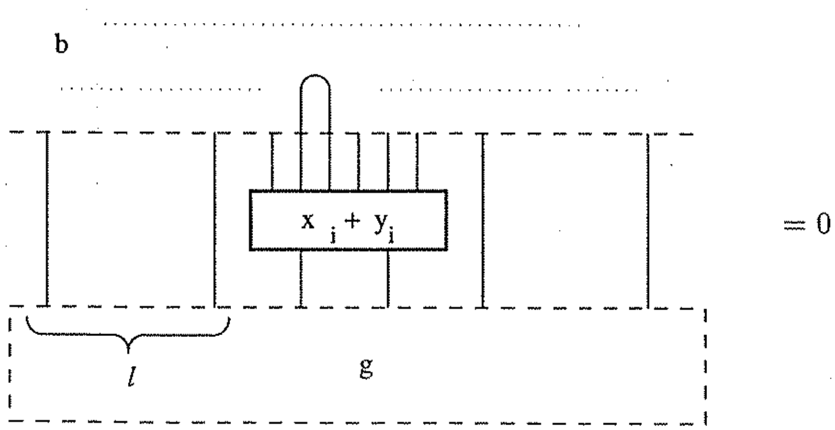
We want to prove that the scalar products of v with dual canonical basis vectors of $V_1^{\otimes n}$ are equal to zero except for one case when the scalar product is $\begin{bmatrix} x_i + y_i \\ x_i \end{bmatrix}$.

So we pick up a vector b from the dual canonical basis of $V_1^{\otimes n}$. To b there is associated a diagram (see Section 3.1) that computes the linear form

$$\psi_b : V_1^{\otimes n} \to \mathbb{C}(q), \quad x \in V_1^{\otimes n} \to (x, b).$$

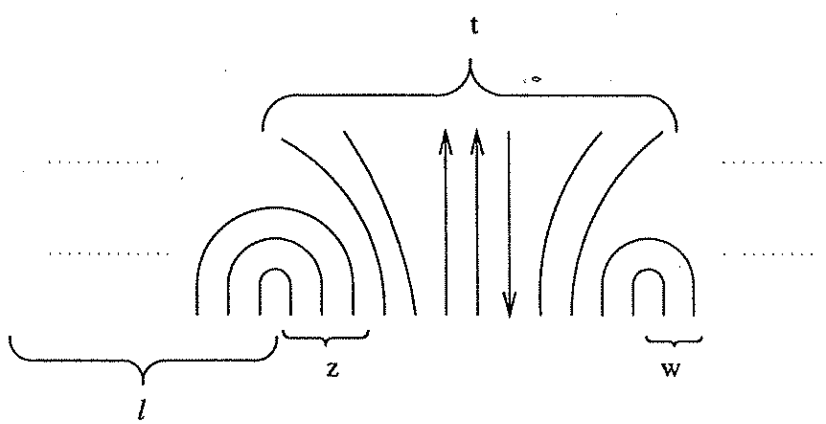
Abusing the notations, from now on we identify the dual canonical vector b and the associated diagram.

If the diagram of b has an arc connecting two lines in $b[l+1, l+x_i+y_i]$, the scalar product (v, b) is zero because then in the diagram for (v, b) there is a simple arc that leaves and enters the top of the projector $p_{x_i+y_i}$ and the diagram evaluates to 0.



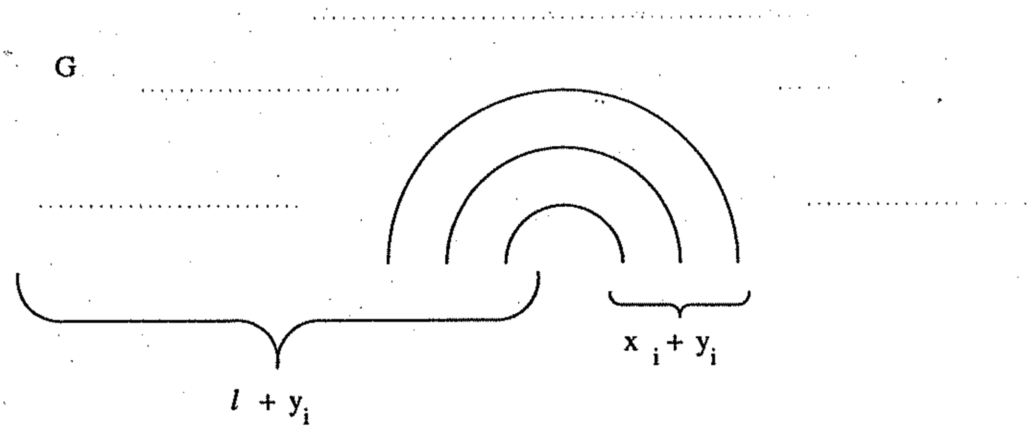
(The covector b is depicted above the punctured line, and is shown schematically by dotted lines except for the above arc.)

Thus, we can disregard this case. Similarly, the scalar product (v, b) is zero if the diagram of b has an arc that connects two lines in $b[l - y_{i-1}, l]$ or two lines in $b[l + x_i + y_i + 1, l + x_i + y_i + x_{i+1}]$ (to see this apply the induction hypothesis for Lemma 3.2 to g). So, we disregard these cases too. Let z be the number of lines that connect $b[l - y_{i-1}, l]$ with $b[l + 1, l + x_i + y_i]$ and w the number of lines connecting $b[l + 1, l + x_i + y_i]$ with $b[l + x_i + y_i + 1, l + x_i + y_i + x_{i+1}]$:

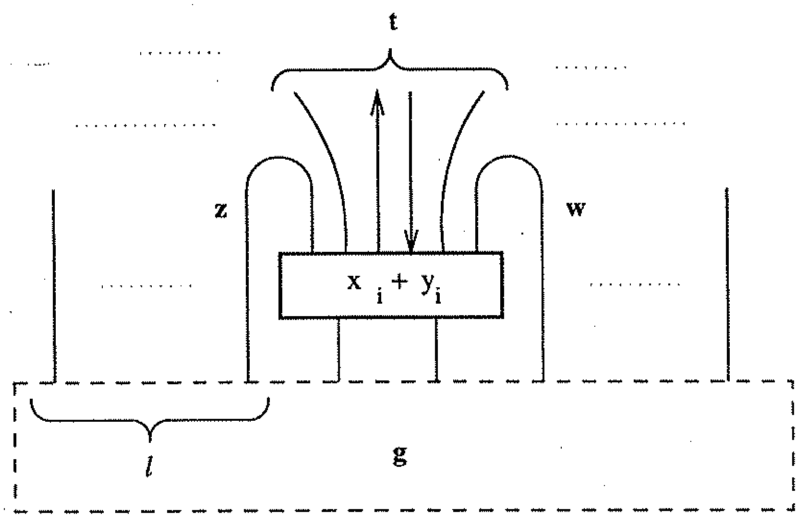


Let $t = x_i + y_i - z - w$. Thus, t is the number of lines attached to the top of the projector $p_{x_i+y_i}$ and which are either arrows or their second endpoint lies in $b[1, l - y_{i-1} - 1]$ or $b[l + x_i + y_i + x_{i+1} + 1, l + x_i + y_i + j]$.

Let G be the dual canonical vector dual to g. Then, because of the way the dual canonical vector is constructed (Chapter 2), there are at least $x_i + y_i$ lines that connect $G[l - y_{i-1} + 1, l + y_i]$ with $G[l + y_i + 1, l + y_i + x_i + x_{i+1}]$:



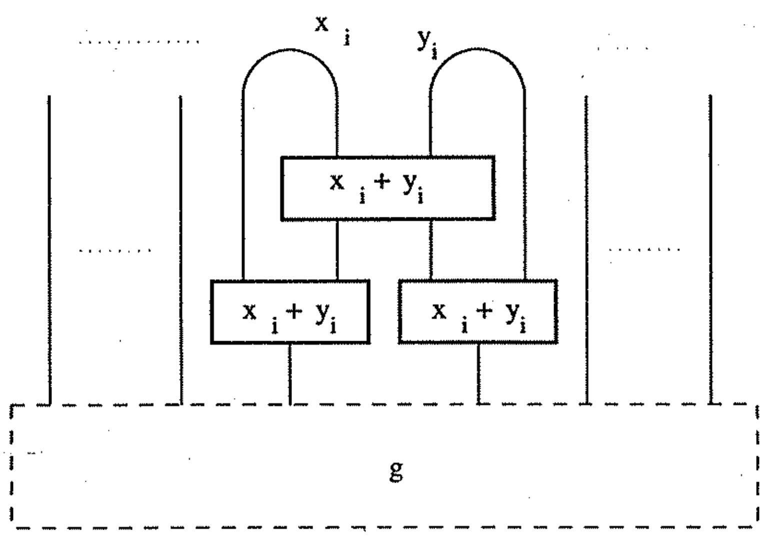
The diagram for (b, v) is schematically depicted below



Now we expand the projector $p_{x_i+y_i}$ into a linear combination of graphical basis vectors of the Temperley-Lieb algebra TL_n (i.e., repeatedly using the identity (4.4)). After that the diagram for (b, v) becomes a linear combination (with certain coefficients) of diagrams that compute scalar products (s, g), where s varies over dual canonical vectors of $V_1^{\otimes n}$. By induction hypothesis only one of these scalar products is non-zero, namely (G, g).

If $t > 0$, then for all the diagrams (s, g) in this sum the number of lines that connect $g[l - y_{i-1} + 1, l + y_i]$ with $g[l + y_i + 1, l + y_i + x_i + x_{i+1}]$ will be less then $x_i + y_i$. Therefore, none of these diagrams represents the scalar product (G, g), and, hence, each of the diagram evaluates to 0. Thus, in this case $(b, v) = 0$.

We are now reduced to the case $t = 0$. In this case the diagram looks schematically as follows.



We now apply lemma 3.1 to get rid of the top projector and then contract the two lower projectors $p_{x_i+y_i}$ back into the canonical basis vector g. That this contraction can be done follows from the induction hypothesis applied to Lemma 3.2. By induction hypothesis we now see that there is only one b in the dual canonical basis of $V_1^{\otimes n}$ such that $(b, v) \neq 0$. The quantum binomial in the formula of part (i) of theorem 1.12 is needed to balance the inverse of the quantum binomial in lemma 3.1.

□

Remark: Suppose we are interested in coefficients of a canonical basis vector $v_{\epsilon_1}\Diamond ... \Diamond v_{\epsilon_n}$ in the product basis $\{v_{\pm 1} \otimes ... \otimes v_{\pm 1}\}$ of $V_1^{\otimes n}$. It it straighforward to rewrite inductive formulas of Theorem 1.12 coordinatewise, using Proposition 1.2 to describe coefficients of the Jones-Wenzl projector in the product basis of $V_1^{\otimes n}$. Then the coordinatewise version of these inductive formulas is exactly the Zelevinsky's recursive formula (see [Z]) for the Kazhdan-Lusztig polynomials in the grassmanian case. Therefore, up to a simple normalization Lusztig's canonical basis coincides with the Kazhdan-Lusztig basis in the grassmannian case. Chapter 5 of this dissertation contains a more intrinsic derivation of this result, without using any explicit formulas for either canonical or Kazhdan-Lusztig bases.

Let us now prove Theorem 1.13. For convenience we state it here again.

THEOREM 1.13. *The element* $v_{a_1-2k_1}\Diamond ... \Diamond v_{a_n-2k_n}$ *of the canonical basis of a tensor product* $V_{a_1}\otimes ... \otimes V_{a_n}$ *is given by*

$$v_{a_1-2k_1}\Diamond ... \Diamond v_{a_n-2k_n} = (\pi_{a_1}\otimes ... \otimes \pi_{a_n})v(a_1-k_1,k_1;...;a_n-k_n,k_n).$$

Proof. Recall that the projection

$$\pi_{a_n}\otimes ... \otimes \pi_{a_1} : V_1^{\otimes(a_n+...+a_1)} \longrightarrow V_{a_n}\otimes ... \otimes V_{a_1}$$

maps each of the dual canonical basis vectors of $V_1^{\otimes(a_n+...+a_1)}$ either to 0 or to an element of the dual canonical basis of $V_{a_n}\otimes ... \otimes V_{a_1}$, and the preimage of a dual canonical basis vector of $V_{a_n}\otimes ... \otimes V_{a_1}$ contains exactly one dual canonical basis vector of $V_1^{\otimes(a_n+...+a_1)}$. We thus obtain a one-to-one map between dual canonical bases of $V_{a_n}\otimes ... \otimes V_{a_1}$ and $V_1^{\otimes(a_n+...+a_1)}$. Denote this map by

$$f: \quad B^{a_n}\heartsuit ... \heartsuit B^{a_1} \rightarrow B_1^{\heartsuit(a_1+...+a_n)}.$$

Then, for $y\in V_{a_1}\otimes ... \otimes V_{a_n}$ and $x\in B^{a_n}\heartsuit ... \heartsuit B^{a_1}$,

$$< y, x > = < (\iota_{a_1}\otimes ... \otimes \iota_{a_n})y, f(x) >$$

This is clear from the graphical representation of the scalar product and the map f (see Chapter 2). Also,

$$(\iota_{a_1}\otimes ... \otimes \iota_{a_n})(\pi_{a_1}\otimes ... \otimes \pi_{a_n})v(a_1-k_1,k_1;...;a_n-k_n,k_n) =$$
$$v(a_1-k_1,k_1;...;a_n-k_n,k_n).$$

This is because $(\iota_{a_1}\otimes...\otimes\iota_{a_n})(\pi_{a_1}\otimes...\otimes\pi_{a_n}) = p_{a_1}\otimes...\otimes p_{a_n}$ and, for $1 \le s \le n$,

$(1_{a_1}\otimes ... \otimes 1_{a_{s-1}} \otimes p_{a_s} \otimes 1_{a_{s+1}} \otimes ... \otimes 1_{a_n})v(a_1-k_1,k_1;...;a_n-k_n,k_n) =$

$v(a_1-k_1,k_1;...;a_n-k_n,k_n)$.

(This equality can be easily checked by induction on n.)

Now, if x is an element of the dual canonical basis of $V_{a_n}\otimes ... \otimes V_{a_1}$, we have

$$< (\pi_{a_1}\otimes ... \otimes \pi_{a_n})v(a_1-k_1,k_1;...;a_n-k_n,k_n), x >=$$

$$< (\iota_{a_1}\otimes ... \otimes \iota_{a_n})(\pi_{a_1}\otimes ... \otimes \pi_{a_n})v(a_1-k_1,k_1;...;a_n-k_n,k_n), f(x) >=$$

$$< v(a_1-k_1,k_1;...;a_n-k_n,k_n), f(x) >=$$

$$= \begin{cases} 1 \text{ if } f(x) \text{ is dual to } v(a_1-k_1,k_1;...;a_n-k_n,k_n) \\ 0 \text{ otherwise} \end{cases}$$

Thus, $(\pi_{a_1}\otimes ... \otimes \pi_{a_n})v(a_1-k_1,k_1;...;a_n-k_n,k_n)$ belongs to the canonical basis of $V_{a_1}\otimes ... \otimes V_{a_n}$. It is easy to see that as k_i varies from 0 to $a_i, i = 1,...,n$, these vectors are linearly independent and, therefore, constitute the canonical basis of $V_{a_1}\otimes ... \otimes V_{a_n}$.

□

3.3. Explicit formulas for canonical bases in invariants of tensor products

Denote the canonical basis of a tensor product $V = V_{a_1}\otimes ... \otimes V_{a_n}$ by $CB(V)$.

The formula for the decomposition of the tensor product (see Section 1.2.4) implies that the space of $U_q(\mathfrak{sl}_2)$-invariants of a tensor product $V = V_{a_1}\otimes ... \otimes V_{a_n}$ is nonzero iff $\sum a_i$ is even and $a_i \le \sum_{j\neq i} a_j$ for all $1 \le i \le n$.

The dual canonical basis of the tensor product $V' = V_{a_n}\otimes ... \otimes V_{a_1}$ intersected with the subspace of $U_q(\mathfrak{sl}_2)$-invariants (relative to $\overline{\Delta}$) of the tensor product defines a basis in this subspace. We call it *the graphical basis of invariants.* On the other hand, the canonical basis of the tensor product $V = V_{a_1}\otimes ... \otimes V_{a_n}$ filters the isotopical decomposition. Precisely, for an element λ of the weight lattice of $U_q(\mathfrak{sl}_2)$ denote by $V[\geq \lambda]$ the submodule $U_q(\mathfrak{sl}_2)V^\lambda$ of V where V^λ is the subspace of weight λ of V. Then $CB(V) \cap V[\geq \lambda]$ defines a basis in $V[\geq \lambda]$ (see Lusztig [L3]). Denote by $V[> 0]$ the sum of $V[\geq \lambda]$ for all $\lambda > 0$. Then the intersection of $CB(V)$ and $V[> 0]$ is a basis in $V[> 0]$. Denote this basis by $CB(V[> 0])$. Note that $V = V[> 0] \oplus \mathrm{Inv}(V)$. Therefore, projecting those vectors of $CB(V)$ that do not belong to the subspace $V[> 0]$ onto $\mathrm{Inv}(V)$ parallel to $V[> 0]$ we obtain a basis in the space of invariants of V. Following Lusztig [L3], we call it *the canonical basis in the space of invariants.* Denote this basis by $CB(\mathrm{Inv}(V))$. Note that

$$< V[> 0], \mathrm{Inv}^{\overline{\Delta}}(V') >= 0 \tag{3.2}$$

This orthogonality relation follows at once if we recall that

$$< xv, w >=< v, \omega(x)w > \tag{3.3}$$

where $x \in \mathbf{U}, v \in V, w \in V'$, the quantum group acts on V via Δ and on V' via $\overline{\Delta}$.

We deduce from (3.2) that

$$< x, b >=< \mathrm{inv}(x), b > \tag{3.4}$$

where x is a vector in V, $\mathrm{inv}(x)$ is its projection onto $\mathrm{Inv}(V)$ parallel to $V[> 0]$ and b is an element of the dual canonical basis of $\mathrm{Inv}^{\overline{\Delta}}(V')$ (where by $\mathrm{Inv}^{\overline{\Delta}}$ we denote invariants with respect to the action given by $\overline{\Delta}$.)

Therefore, the canonical basis $CB(\mathrm{Inv}(V))$ is given by vectors $\mathrm{inv}(x)$ for x from the canonical basis $CB(V)$ such that $x \notin V[> 0]$.

Let us first consider the case when V is a tensor power of V_1, i.e., $a_1 = a_2 = ... = a_n = 1$. Then $\mathrm{Inv}(V_1^{\otimes n}) \neq 0$ if and only if n is even. Let $k = \frac{n}{2}$. From the theorem 1.12 we know that $x \in CB(V)$ is obtained by applying a composition of Jones-Wenzl projectors to the vector $v_{-1}^{\otimes a} \otimes v_1^{\otimes(2k-a)}$ and then scaling by a product of quantum binomials. Denote by cp_x the operator $V_1^{\otimes 2k} \to V_1^{\otimes 2k}$ given by this particular composition of Jones-Wenzl projectors and scaling. Thus, the canonical basis vector

$$x = \mathrm{cp}_x(v_{-1}^{\otimes a} \otimes v_1^{\otimes 2k-a}).$$

Note that $\mathrm{inv}(x) \neq 0$ implies that x is a vector of weight 0, that is, $a = k$. Because Jones-Wenzl projectors intertwine the action of the quantum group $\mathbf{U}$, we have

$$\mathrm{inv}(\mathrm{cp}_x(v_{-1}^{\otimes k} \otimes v_1^{\otimes k})) = \mathrm{cp}_x(\mathrm{inv}(v_{-1}^{\otimes k} \otimes v_1^{\otimes k}))$$

Recall that in the dual case of the action via $\overline{\Delta}$, the closure construction, explained in the beginning of Section 2.4, gives explicit formulas (in the product basis of $V_1^{\otimes 2k}$) of the $\mathbf{U}$-module projection $V_1^{\otimes 2k} \to \mathrm{Inv}(V_1^{\otimes 2k})$.

Of course, the same construction (with q changed to q^{-1}) works in our case of the action via Δ. In fact, for our purposes here we only need to know $\mathrm{inv}(v_{-1}^{\otimes k} \otimes v_1^{\otimes k})$ explicitly. It is given by the following

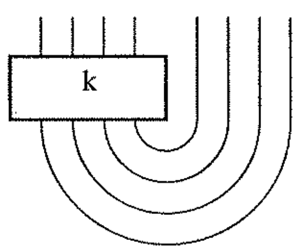
PROPOSITION 3.3.

$$\begin{aligned}(3.5)\qquad & \mathit{inv}(v_{-1}^{\otimes k} \otimes v_1^{\otimes k}) = \\ & (-1)^k [k+1]^{-1} (p_k \otimes 1^{\otimes k})(1^{\otimes (k-1)} \otimes \delta_1^{\Delta} \otimes 1^{\otimes (k-1)}) ... (1 \otimes \delta_1^{\Delta} \otimes 1) \delta_1^{\Delta}(1)\end{aligned}$$

where δ_1^{Δ} *is the intertwiner* $V_0 \to V_1 \otimes V_1$ *relative to the action given by* Δ,

$$\delta_1^{\Delta}(1) = v_1 \otimes v_{-1} - q v_{-1} \otimes v_1$$

Graphically, the R.H.S. of (3.5) is given by the diagram



Finally for a complete description of canonical basis in the invariants of $V_1^{\otimes 2k}$ we need to know for which x from the canonical basis of $V_1^{\otimes 2k}$ the projection $\mathrm{inv}(x) \neq 0$. Using our description of the dual canonical basis in $\mathrm{Inv}^{\overline{\Delta}}$ and canonical – dual canonical duality, we immediately obtain

PROPOSITION 3.4. *Let x be a canonical basis vector in $V_1^{\otimes 2k}$,*

$$x = v_{\epsilon_1} \Diamond \ldots \Diamond v_{\epsilon_{2k}}, \qquad \epsilon_i \in \{1, -1\}, 1 \leq i \leq 2k.$$

Then $inv(x) \neq 0$ if and only if $\sum_{i=1}^{2k} \epsilon_i = 0$ and for any $i, 1 \leq i \leq 2k$ we have $\sum_{j=1}^{i} \epsilon_j \leq 0$.

Remark: Notice that the number of sequences of length $2k$ of ones and negative ones that satisfy $\sum_{i=1}^{2k} \epsilon_i = 0$ and $\sum_{j=1}^{i} \epsilon_j \leq 0$ for $i = 1, \ldots, 2k$ is equal to the k-th Catalan number $\frac{1}{k+1}\begin{pmatrix} 2k \\ k \end{pmatrix}$ which is the dimension of the Temperley-Lieb algebra TL_k.

More generally, for an arbitrary tensor product $V_{a_1} \otimes \ldots \otimes V_{a_n}$ the canonical basis vectors in the invariants of this tensor product are given by projecting certain canonical basis vectors in the invariants of $V_1^{\otimes(a_1+\ldots+a_n)}$ via $\pi_{a_1} \otimes \ldots \otimes \pi_{a_n}$. This construction is completely analogous to the one stated in Theorem 1.13 and is left to the reader as an exercise.

CHAPTER IV

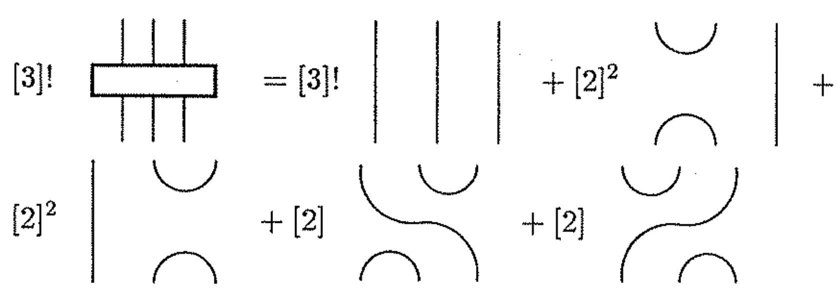
THE POSITIVE INTEGRAL STRUCTURE OF JONES-WENZL PROJECTOR

4.1. Positive integral decomposition of the projector

The Jones-Wenzl projector p_n is an element of the Temperley-Lieb algebra TL_n. We are interested in the coefficients of p_n in the dual canonical basis B_n^{TL} of the Temperley-Lieb algebra (this basis was defined in Section 1.2.3). Decompose $[n]!p_n$ as a linear combination of vectors of B_n^{TL}:

$$[n]!p_n = \sum_{d \in B_n^{TL}} P(d)d \tag{4.1}$$

For example,



For $s \in \mathrm{S}_n$ and $d \in B_n^{TL}$ denote by $R(s,d)$ the coefficient of d in the decomposition of $T(s)$ as a linear combination of dual canonical basis vectors in TL_n:

$$T(s) = \sum_{d \in B_n^{TL}} R(s,d)d \tag{4.2}$$

where $T(s)$ is given by (1.25),(1.26). Denote by $l(s)$ the length of the permutation s.

THEOREM 4.1. *For any $n \in \mathbb{N}$ and $d \in B_n^{TL}$ the coefficient $P(d)$ belongs to $q^{\frac{n(n-1)}{2}}\mathbb{N}[q^{-1}]$.*

THEOREM 4.2. *For any $s \in \mathbf{S}_n$ and $d \in B_n^{TL}$ the coefficient $R(s,d)$ belongs to $q^{\frac{l(s)}{2}}\mathbb{N}[q^{-1}]$.*

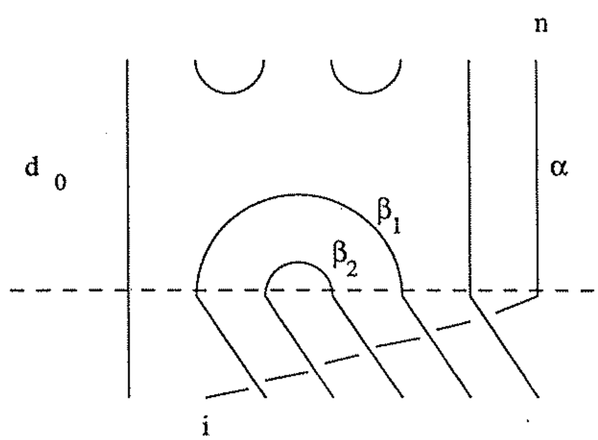
Theorem 4.1 follows immediately from Theorem 4.2 and Theorem 1.4.

Proof of theorem 4.2: We use induction on n. The induction base $n = 1$ is obvious.

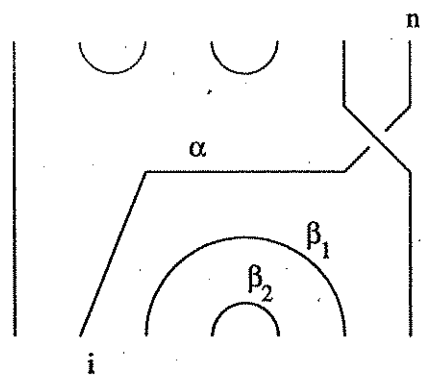
Induction step: Suppose that for any $s \in \mathbf{S}_{n-1}$ and $d \in B_{n-1}^{TL}$ the coefficient $R(s,d) \in q^{\frac{l(s)}{2}}\mathbb{N}[q^{-1}]$. Take $s \in \mathbf{S}_n$. We have $s = s's''$ where $s' \in \mathbf{S}_{n-1}$ and $s'' = s_{n-1}...s_{i+1}s_i$ for some $i, 1 \leq i \leq n$ (if $i = n$ then $s'' = 1$). Hence,

$$T(s) = T(s')T(s'') = T(s')T_{n-1}...T_i$$

Because all coefficients of $T(s')$ in the basis B_{n-1}^{TL} are in $q^{\frac{l(s')}{2}}\mathbb{N}[q^{-1}]$, it is enough to prove that for any $d_0 \in B_{n-1}^{TL}$ all coefficients of $d_0T_{n-1}...T_i$ in the basis B_n^{TL} are in $q^{\frac{n-i}{2}}\mathbb{N}[q^{-1}]$. The element $d_0T_{n-1}...T_i$ is schematically depicted below.



This picture consists of n arcs that join $2n$ points (n top and n bottom points). These arcs have $n - i$ intersection points. Denote by α the arc that connects the rightmost top point with the i-th (counted from the left) bottom point. Denote by $\beta_1, ..., \beta_t$ those of the remaining $n-1$ arcs that intersect α twice. We isotop $\beta_1, ..., \beta_t$ so that they do not intersect α and lie beneath α.



After that isotopy each of $n-1$ arcs intersects α in no more than one point. That all coefficients of $d_0 T_{n-1} ... T_i$ belong to $q^{\frac{n-i}{2}}\mathbb{N}[q^{-1}]$ follows now from the obvious fact that all coefficients of $T_{j-1} ... T_1$ in the dual canonical basis B_j^{TL} belong to $q^{\frac{j-1}{2}}\mathbb{N}[q^{-1}]$. □

4.2. Reduction formulas for the coefficients of the projector

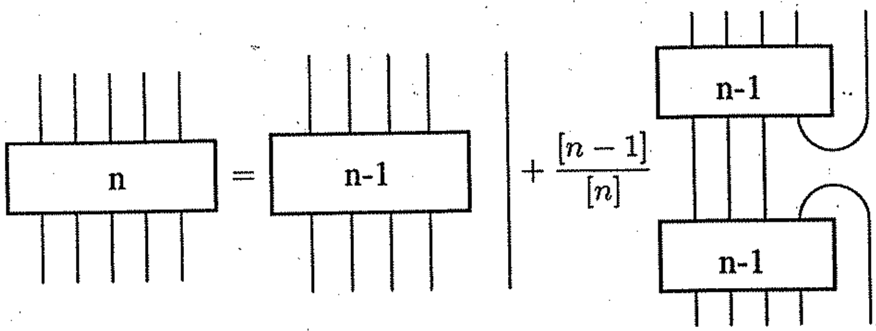
In this section we give new formulas for the Jones-Wenzl projector and its coefficients. These formulas can be used for a simple derivation of various formulas for Racah-Wigner and Clebsch-Gordan coefficients. We start with the following inductive definition, due to Wenzl (see [KaL] and references therein), of the Jones-Wenzl projector p_n:

THEOREM 4.3. *The Jones-Wenzl projector $p_n \in TL_n$ satisfies*

$$p_1 = 1, \qquad (4.3)$$

$$p_n = p_{n-1} + \frac{[n-1]}{[n]} p_{n-1} U_{n-1} p_{n-1}.$$

In terms of diagrams, (4.3) says

(4.4) 

Jones-Wenzl projectors are also characterized by

THEOREM 4.4(SEE [KAL]). *The elements p_n satisfy*

(i) $p_n^2 = p_n, \quad n \geq 1,$

(ii) $p_n U_i = 0, \quad 1 \leq i \leq n-1,$

(iii) $U_i p_n = 0, \quad 1 \leq i \leq n-1.$

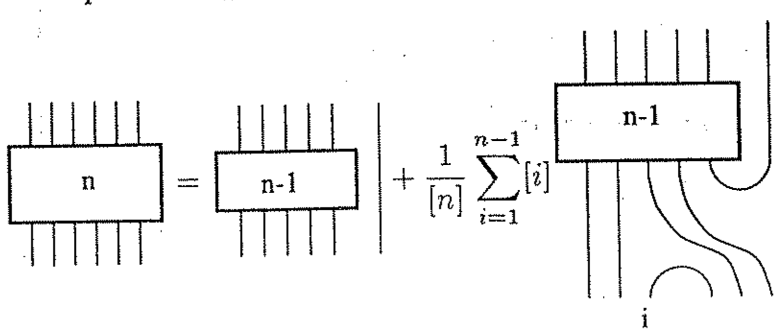
Properties (i),(ii) or (i),(iii) uniquely determine p_n.

In this paragraph we obtain an inductive formula for the coefficients of the Jones-Wenzl projector p_n in the dual canonical basis B_n^{TL} of TL_n. This formula refines Wenzl formula (4.3). Notice that the last summand in the Wenzl formula contains two projectors of size $n-1$ connected by $n-2$ lines. If we expand one of these projectors, most of the terms vanish after composing with the other projector. It turns out that the remains are easy to compute and we get

THEOREM 4.5. *For $n > 1$ the Jones-Wenzl projector decomposes*

$$p_n = \frac{1}{[n]} p_{n-1} \left(\sum_{i=1}^{n} [i] U_{n-1} U_{n-2} \ldots U_i\right) \tag{4.5}$$

where for $i = n$ the product $U_{n-1} \ldots U_i$ is equal to 1. In terms of diagrams,

(4.6) 

Note that this is equivalent to the formula

$$(4.7) \qquad p_n = \frac{1}{[n]!}\prod_{j=1}^{n}\sum_{i=1}^{j}[i]U_{j-1}U_{j-2}...U_i$$

Proof of Theorem 4.5: Expand

$$\frac{1}{[n]}p_{n-1}(\sum_{i=1}^{n}[i]U_{n-1}U_{n-2}...U_i)$$

in the dual canonical basis of the Temperley-Lieb algebra. Notice that the coefficient of the unit diagram in this expansion is equal to 1. Thus, by Theorem 4.4, it is enough to prove that for $k = 1, ..., n-1$,

$$p_{n-1}(\sum_{i=1}^{n}[i]U_{n-1}U_{n-2}...U_i)U_k = 0$$

We use the following easy

LEMMA 4.6.

(a) For any $k = 1, ..., n-2$ there are elements $x, y \in TL_n$ such that

$$(\sum_{i=1}^{n}[i]U_{n-1}U_{n-2}...U_i)U_k = U_{k-2}x + U_k y$$

(b) There is $x \in TL_n$ such that

$$(\sum_{i=1}^{n}[i]U_{n-1}U_{n-2}...U_i)U_{n-1} = U_{n-3}x$$

This lemma follows from the Temperley-Lieb algebra relations and the identity $[2][k] = [k-1] + [k+1]$. □

Now, by this lemma, for $k = 1, ..., n-2$,

$$p_{n-1}(\sum_{i=1}^{n}[i]U_{n-1}U_{n-2}...U_i)U_k = p_{n-1}(U_{k-2}x + U_k y)$$

(for some $x, y \in TL_n$)

$$= (p_{n-1}U_{k-2})x + (p_{n-1}U_k)y = 0.$$

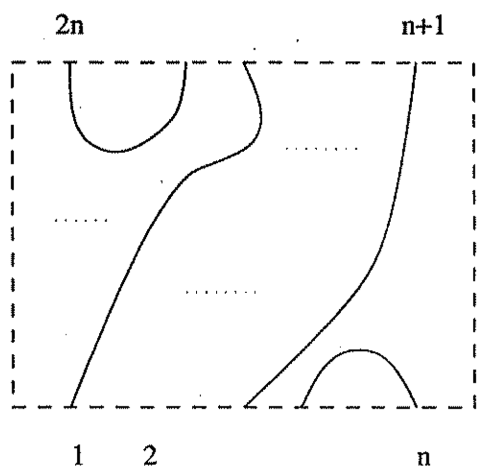
Also,

$$p_{n-1}(\sum_{i=1}^{n}[i]U_{n-1}U_{n-2}...U_i)U_{n-1} = p_{n-1}U_{n-3}x = 0$$

The proof of Theorem 4.5 is finished. □

To apply Theorem 4.5 to the computations of coefficients of Jones-Wenzl projector we give below an equivalent statement.

First a few more notations. A diagram depicting an element of the dual canonical basis of TL_n consists of n arcs connecting $2n$ points on the boundary of a rectangle: n point on the top and n points on the bottom. Numerate these points by $1, 2, ..., 2n$ in the anticlockwise order starting from the lower left corner:



By abuse of notations we identify a diagram with the associated element of the dual canonical basis of TL_n. An arc of a diagram is called *tiny* if it connects two points numbered consequently (1 follows after $2n$). For a diagram $d \in B_n^{TL}$ define *the set of lower tiny arcs* $L(d)$ as a subset of $\{1, 2, ..., n\}$ with $i \in L(d)$ if and only if d contains an arc connecting i and $i+1$. Note that for any diagram d the set of lower tiny arcs is non-empty.

Take a diagram $d \in B_n^{TL}$. The diagram d is described by decomposing the set $\{1, 2, ..., 2n\}$ into n pairs so that the n arcs joining two elements of each pair do not intersect. Pick an $i \in L(d)$. There is an arc connecting points numbered i and $i+1$. Delete the pair of points $i, i+1$ and renumerate the rest of points (points

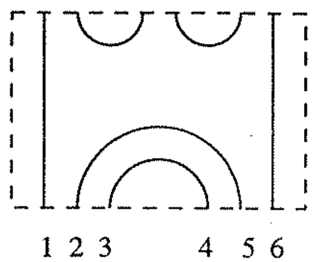
$\{1, \ldots, i-1, i+2, \ldots, 2n\}$) as follows:

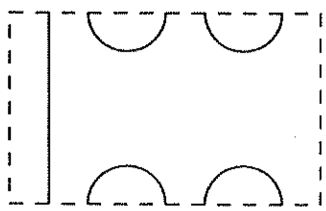
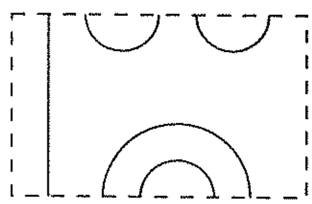
$$1 \to 1, \ldots, i-1 \to i-1, i+2 \to i, \ldots, 2n \to 2n-2.$$

Then the splitting of $\{1, \ldots, 2n\}$ into n pairs associated to the diagram d restricts to a splitting of $\{1, \ldots, 2n-2\}$ into $n-1$ pairs. This splitting defines a diagram because no two arcs intersect.

We call this new diagram *the reduction of diagram d at i* and denote it by $d \sim i$.

Example:

$d =$ 

$d \sim 3 =$  $d \sim 6 =$ 

COROLLARY 4.7. *For any diagram d*

$$P(d) = \sum_{i \in L(d)} P(d \sim i)[i], \tag{4.8}$$

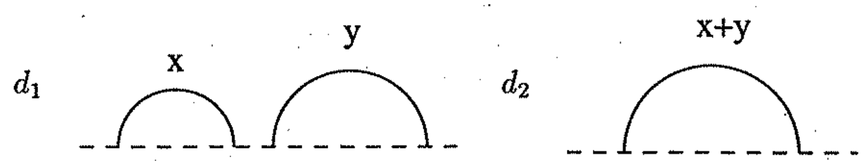
where $P(d)$ is defined in Section 4.1 and $L(d)$ is the set of lower tiny arcs of d as defined above.

Corollary 4.7 is equivalent to Theorem 4.5.

Remark. The group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ of symmetries of a rectangle acts on the set B_n^{TL} of diagrams of size n. For any diagram $d \in B_n^{TL}$ the coefficient $P(d)$ is equal to the coefficient $P(d_1)$ of any diagram d_1 in the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-orbit of d.

Recall that a line marked by n denotes n lines going in parallel.

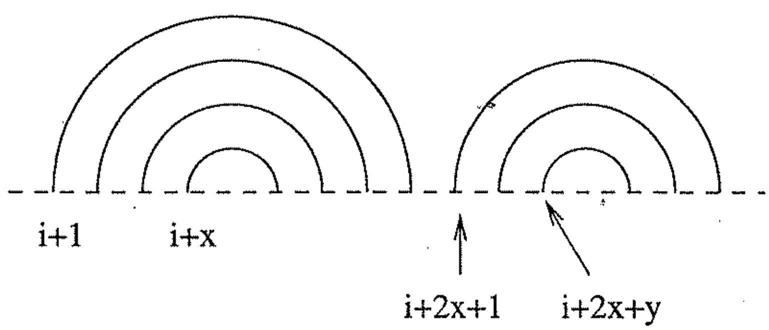
PROPOSITION 4.8. *Let d_1, d_2 be two diagrams from B_n^{TL} that differ as shown below.*



They are the same outside the part that is shown. Then

$$P(d_1) = \begin{bmatrix} x+y \\ x \end{bmatrix} P(d_2). \tag{4.9}$$

Proof. Let the leftmost point shown on the above picture have number $i+1$



Introduce a partial order $\leq$ on the set of triples (a, b, c) with $a, b, c \in \mathbb{N}$: $(a, b, c) \leq (a_1, b_1, c_1)$ iff $a \leq a_1$ or $a = a_1, b \leq b_1$ or $a = a_1, b = b_1, c \leq c_1$. The proof is by induction on (n, x, y) where n is the size of the diagram d_1. The induction base is the case $x = 0$ or $y = 0$ and is obvious.

Induction step: Note that the set of lower tiny arcs $L(d_1), L(d_2)$ of diagrams d_1, d_2 differ only by elements $i + x, i + 2x + y, i + x + y$:

$$\begin{aligned} L(d_1) &= \{i + x\} \cup \{i + 2x + y\} \cup (L(d_1) \cap L(d_2)), \\ L(d_2) &= \{i + x + y\} \cup (L(d_1) \cap L(d_2)). \end{aligned} \tag{4.10}$$

Therefore,

$$(4.11)$$

$$P(d_1) = \sum_{j\in L(d_1)} [j]P(d_1 \sim j) =$$

$$[i+x]P(d_1 \sim i+x) + [i+2x+y]P(d_1 \sim i+2x+y) + \sum_{j\in L(d_1)\cap L(d_2)} [j]P(d_1 \sim j) =$$

$$[i+x]\begin{bmatrix} x+y-1 \\ x-1 \end{bmatrix} P(d_2 \sim i+x+y) + [i+2x+y]\begin{bmatrix} x+y-1 \\ y-1 \end{bmatrix} P(d_2 \sim i+x+y)+$$

$$\begin{bmatrix} x+y \\ y \end{bmatrix} \sum_{j\in L(d_1)\cap L(d_2)} [j]P(d_2 \sim j) =$$

$$[x+y+i]\begin{bmatrix} x+y \\ y \end{bmatrix} P(d_2 \sim i+x+y) + \begin{bmatrix} x+y \\ y \end{bmatrix} \sum_{j\in L(d_1)\cap L(d_2)} [j]P(d_2 \sim j) =$$

$$\begin{bmatrix} x+y \\ y \end{bmatrix} \sum_{j\in L(d_2)} [j]P(d_2 \sim j) = \begin{bmatrix} x+y \\ y \end{bmatrix} P(d_2).$$

The third equality uses the induction hypothesis

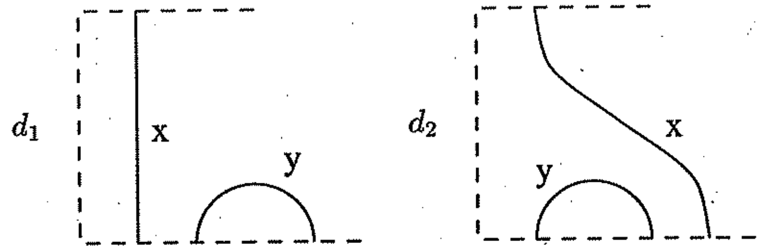
$$\begin{aligned} & P(d_1 \sim i+x) = \begin{bmatrix} x-1+y \\ x-1 \end{bmatrix} P(d_2 \sim i+x+y), \\ (4.12) \qquad & P(d_1 \sim i+2x+y) = \begin{bmatrix} x+y-1 \\ y-1 \end{bmatrix} P(d_2 \sim i+x+y), \\ & P(d_1 \sim j) = \begin{bmatrix} x+y \\ y \end{bmatrix} P(d_2 \sim j), \text{ where } j \in L(d_1)\cap L(d_2) \end{aligned}$$

In the fourth equality we use the identity

$$(4.13) \qquad [i+x]\begin{bmatrix} x+y-1 \\ x-1 \end{bmatrix} + [i+2x+y]\begin{bmatrix} x+y-1 \\ y-1 \end{bmatrix} = [x+y+i]\begin{bmatrix} x+y \\ y \end{bmatrix}.$$

□

PROPOSITION 4.9. *Let d_1, d_2 be two diagrams from B_n^{TL} that differ as depicted below.*



They are the same outside the part that is shown. Then

$$P(d_1) = \begin{bmatrix} x+y \\ x \end{bmatrix} P(d_2). \tag{4.14}$$

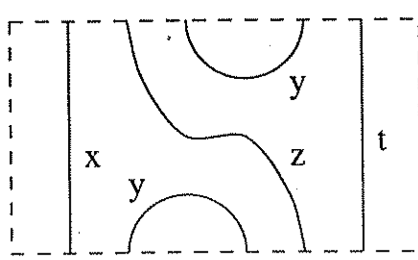
Proof is similar to the proof of the previous proposition and is done by induction on the triple (n, x, y) with respect to the same partial order. We have

$$L(d_1) = \{x+y\} \cup (L(d_1) \cap L(d_2)), \quad L(d_2) = \{y\} \cup (L(d_1) \cap L(d_2))$$

Therefore,

$$\begin{aligned}
&P(d_1) = \sum_{j \in L(d_1)} [j] P(d_1 \sim j) = \\
&[x+y] P(d_1 \sim x+y) + \sum_{j \in L(d_1) \cap L(d_2)} [j] P(d_1 \sim j) = \\
&[x+y] \begin{bmatrix} x+y-1 \\ y-1 \end{bmatrix} P(d_2 \sim y) + \sum_{j \in L(d_1) \cap L(d_2)} [j] \begin{bmatrix} x+y \\ y \end{bmatrix} P(d_2 \sim j) = \\
&\begin{bmatrix} x+y \\ y \end{bmatrix} [y] P(d_2 \sim y) + \begin{bmatrix} x+y \\ y \end{bmatrix} \sum_{j \in L(d_1) \cap L(d_2)} [j] P(d_2 \sim j) = \\
&\begin{bmatrix} x+y \\ y \end{bmatrix} \sum_{j \in L(d_2)} [j] P(d_2 \sim j) = \begin{bmatrix} x+y \\ y \end{bmatrix} P(d_2).
\end{aligned} \tag{4.15}$$

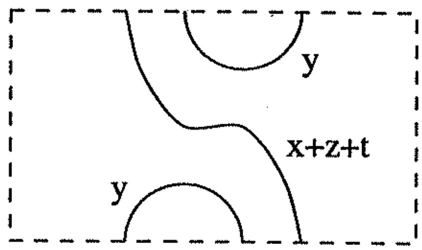
PROPOSITION 4.10. *Let d be a diagram*



where x, y, z, t are non-negative integers satisfying $x+y+z+t \geq 1$. Then

$$P(d) = \begin{bmatrix} x+y \\ y \end{bmatrix} \begin{bmatrix} t+y \\ y \end{bmatrix} [y]![x+z+t+y]! \tag{4.16}$$

Proof. By Proposition 4.9 $P(d) = \begin{bmatrix} x+y \\ y \end{bmatrix} \begin{bmatrix} t+y \\ y \end{bmatrix} P(d_1)$ where diagram d_1 is



Corollary 4.7 implies $P(d_1) = [y]![x+y+z+t]!$

□

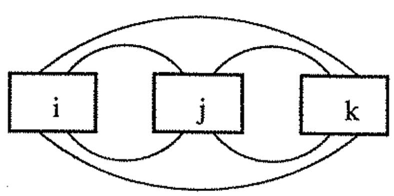
4.3. Factorization of Racah-Wigner coefficients

We first recall the formula for the theta-curve, as in [KaL]. A triple of nonnegative integers i, j, k is called admissible if

$$i+j+k = 0(\text{mod } 2),$$

$$i+j-k \geq 0, i+k-j \geq 0, j+k-i \geq 0.$$

DEFINITION 4.1. *For an admissible triple (i, j, k) the theta-curve is the following diagram*



Let $\theta(i, j, k)$ be the value of this diagram.

THEOREM 4.11 (SEE [KAL]). *For an admissible triple (i, j, k)*

$$\theta(i,j,k) = \frac{(-1)^{\frac{i+j+k}{2}} [\frac{i+j+k}{2}+1]![\frac{i+j-k}{2}]![\frac{i+k-j}{2}]![\frac{j+k-i}{2}]!}{[i]![j]![k]!}. \square \tag{4.17}$$

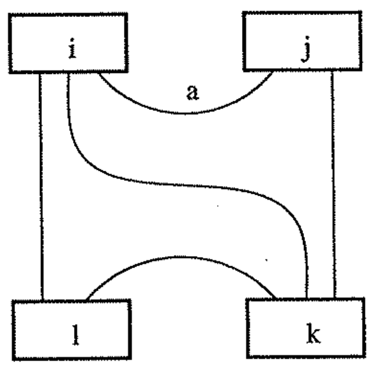
Fix four non-negative integers i, j, k, l such that $i+j+k+l$ is even. In what follows we suppose, without loss of generality, that $i+k \geq j+l$.

We consider the space $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_l \otimes V_k, V_i \otimes V_j)$ which we denote for simplicity by $V_{i,j,k,l}$. This space can be naturally identified with $\mathrm{Inv}_{U_q(\mathfrak{sl}_2)}(V_i \otimes V_j \otimes V_k \otimes V_l)$, that is the space of invariants of a certain tensor product. Thus, in $\mathrm{Inv}_{U_q(\mathfrak{sl}_2)}(V_i \otimes V_j \otimes V_k \otimes V_l)$ the dual canonical basis is defined. We will call the image of this basis under the natural isomorphism

$$\mathrm{Inv}_{U_q(\mathfrak{sl}_2)}(V_i \otimes V_j \otimes V_k \otimes V_l) \quad \rightarrow \quad \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_l \otimes V_k, V_i \otimes V_j)$$

the dual canonical basis in $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_l \otimes V_k, V_i \otimes V_j)$.

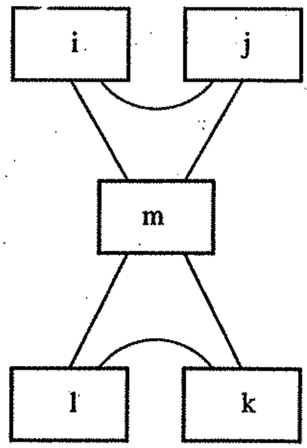
Denote the diagram



by $B_a(i,j,k,l)$. Number a satisfies the following inequalities

$$a \leq j, a \leq \frac{i+j+l-k}{2}, a \geq \frac{i+j-k-l}{2} \tag{4.18}$$

Diagrams $B_a(i,j,k,l)$ are in a one-to-one correspondence with elements of the dual canonical basis of $V_{i,j,k,l}$. For all m such that (i,j,m) and (m,k,l) are admissible denote by $T_m(i,j,k,l)$ the following element of $V_{i,j,k,l}$



Vectors $\{T_m(i,j,k,l)\}_m$ form a basis of the space $V_{i,j,k,l}$. The transition matrix from the basis $\{T_m(i,j,k,l)\}_m$ to the basis $\{B_a(i,j,k,l)\}_a$ is upper-triangular with units on the diagonal.

Let us compute transition coefficients from the basis $\{T_m(i,j,k,l)\}_m$ to $\{B_a(i,j,k,l)\}_a$. This is simply a restatement of Proposition 4.10.

PROPOSITION 4.12. *For i,j,k,l,m such that $i+k \geq j+l$, (i,j,m) and (m,k,l) are admissible one has*

$$T_m(i,j,k,l) = \sum_a \mathcal{A}_{m,a}(i,j,k,l) B_a(i,j,k,l), \tag{4.19}$$

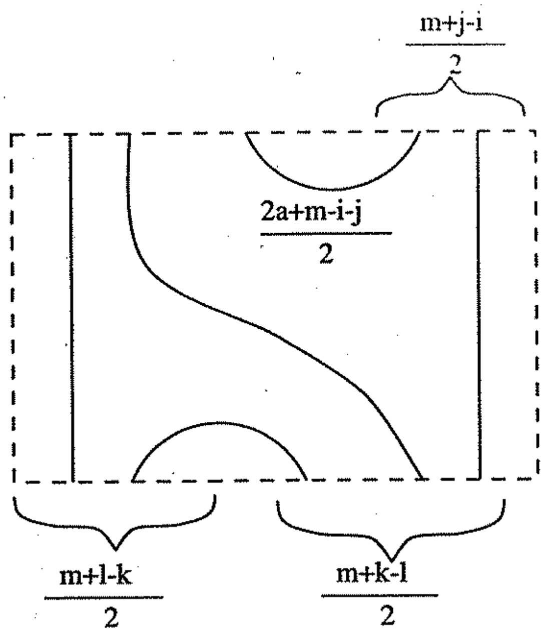
where

$$\mathcal{A}_{m,a}(i,j,k,l) = \frac{[\frac{m+j-i}{2}]![\frac{m+l-k}{2}]![\frac{i+j+m}{2} - a]!}{[a + \frac{m-i-j}{2}]![j-a]![\frac{i+j+l-k}{2} - a]![m]!}. \tag{4.20}$$

The sum is over those a that satisfy the inequalities

$$a \leq j, a \leq \frac{i+j+l-k}{2}, a \geq \frac{i+j-m}{2}. \tag{4.21}$$

Proof. Let d be the diagram



Then $\mathcal{A}_{m,a}(i,j,k,l) = \frac{1}{[m]!}P(d)$ and $P(d)$ is computed using proposition 4.10. □

For $i+k < j+l$ everything is as before if we substitute $i \leftrightarrow j, k \leftrightarrow l$.

Next we compute coefficients of the inverse matrix – transition matrix from the basis $\{B_a(i,j,k,l)\}_a$ to $\{T_m(i,j,k,l)\}_m$.

PROPOSITION 4.13. *For non-negative integers i,j,k,l,a that satisfy (4.18) and*

$$i+j+k+l = 0(mod\ 2), \quad i+k \geq j+l \tag{4.22}$$

we have

$$B_a(i,j,k,l) = \sum_m \mathcal{F}_{am}(i,j,k,l) T_m(i,j,k,l), \tag{4.23}$$

where

$$\mathcal{F}_{a,m}(i,j,k,l) = \frac{(-1)^{a+\frac{i+j-m}{2}}[m+1]![j-a]![\frac{i+j+l-k}{2}-a]!}{[\frac{m+l-k}{2}]![\frac{m+i+j}{2}-a+1]![\frac{m+j-i}{2}]![\frac{i+j-m}{2}-a]!} \tag{4.24}$$

and the sum is over those m such that $(m, i-a, j-a)$ is admissible and $m \geq k-l$.

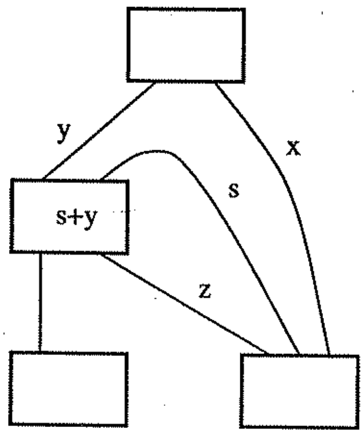
Proof. First we will need the following lemmas

Lemma 4.14 (see [KaL]).

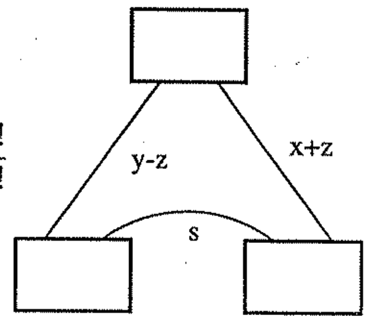
$$B_0(x,y,y,x) = \sum_z \frac{(-1)^z[z+1]}{\theta(x,y,z)} T_z(x,y,y,x), \tag{4.25}$$

where the sum is over those z such that the triple (x,y,z) is admissible. □

Lemma 4.15.



$$= \frac{[y]![s+y-z]!}{[y-z]![s+y]!}$$

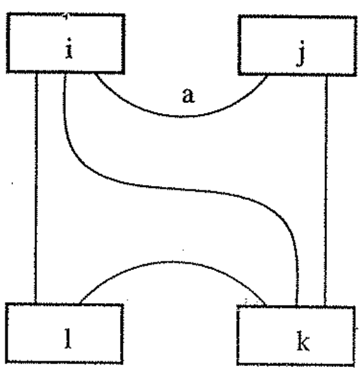


if $y \geq z$, otherwise (if $y < z$) the L.H.S. is equal to 0.

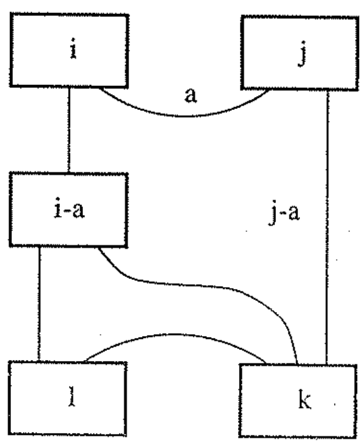
Lemma 4.15 is the same as lemma 2 in [MV]. It is also an immediate corollary of Proposition 4.10. □

We can now derive (4.23):

$G_a(i,j,k,l) =$

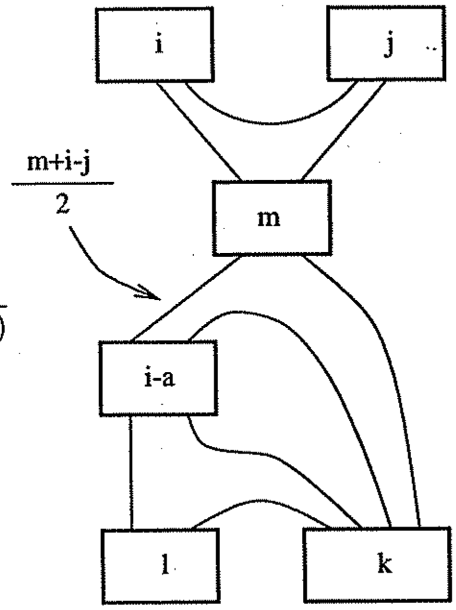


$=$



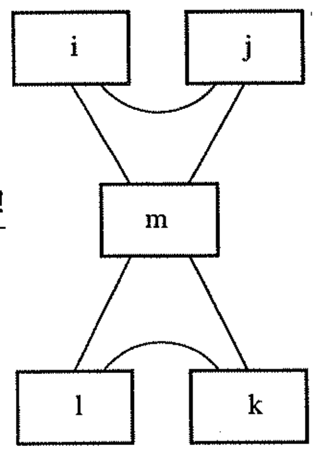
$\overset{\text{Lemma 4.14}}{=}$

$$= \sum_{m,(m,i-a,j-a)-\text{admiss.}} \frac{(-1)^m[m+1]}{\theta(m,i-a,j-a)}$$



Lemma 4.15
$=$

$$= \sum_m \frac{(-1)^m[m+1][\frac{m+i-j}{2}]![\frac{i+j+l-k}{2}-a]!}{\theta(m,i-a,j-a)[\frac{m+l-k}{2}]![i-a]!}$$

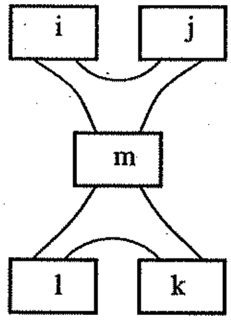


where the last sum is over those m such that $(m,i-a,j-a)$ is admissible and $m \geq k-l$.□

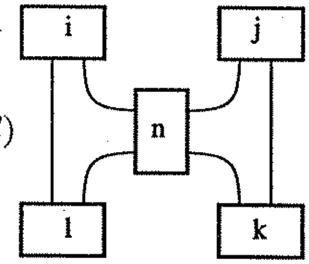
Note that the matrices $\mathcal{A}(i,j,k,l)$ and $\mathcal{F}(i,j,k,l)$ are the inverses of each other:

(4.26) $$\sum_a \mathcal{A}_{m,a}(i,j,k,l)\mathcal{F}_{a,n}(i,j,k,l) = \delta_{m,n}.$$

Define the $6j$-symbol $W_{m,n}(i,j,k,l)$ by



$$= \sum_n W_{m,n}(i,j,k,l)$$



Propositions 4.12,4.13 give a factorization of $6j$-symbols:

PROPOSITION 4.16. *For m, n, i, j, k, l such that the triples $(i,j,m), (m,k,l)$, $(i,n,l), (j,k,n)$ are admissible and $i+k \geq j+l$ we have*

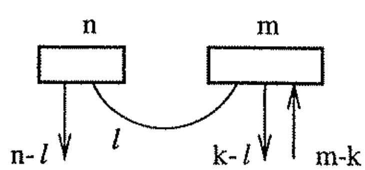
$$W_{m,n}(i,j,k,l) = \sum_a \mathcal{A}_{m,a}(i,j,k,l)\mathcal{F}_{j-a,n}(k,j,i,l) \tag{4.27}$$

where the sum is over those a for which the R.H.S. of (4.27) is well-defined.

Together Propositions 4.12,4.13,4.16 compute the $6j$-symbol.

4.4. Factorization of Clebsch-Gordan coefficients

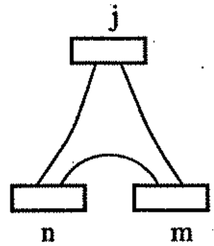
The vector $v^{2l-n} \heartsuit v^{m-2k}$ of the dual canonical basis of $V_n \otimes V_m$ is given by the following diagram



(for $k \geq l$, similarly for $k < l$.) Using the graphical identity

to remove all closed arcs from this diagram we immediately obtain Proposition 1.6(ii).

Clebsch-Gordan coefficients are matrix coefficients of the intertwiner

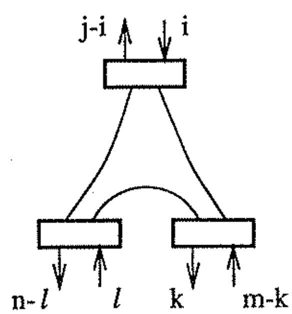
$v_j^{nm} =$ 

They are given by

$$< v_j^{nm}(v^{2l-n} \otimes v^{m-2k}), v^{j-2i} > \cdot < v^{j-2i}, v^{j-2i} >^{-1}$$

for $0 \leq l \leq n, 0 \leq k \leq m, 0 \leq i \leq j$. Note that $< v^{j-2i}, v^{j-2i} >^{-1} = \begin{bmatrix} j \\ i \end{bmatrix}$ and the complexity is hidden in the first term of the above product. Clebsch-Gordan coefficients for $U_q(\mathfrak{sl}_2)$ were originally computed in [KR].

The scalar product $< v_j^{nm}(v^{2l-n} \otimes v^{m-2k}), v^{j-2i} >$, which we also denote $C^j_{nm}(l,k,i)$, is equal to the evaluation of the following diagram

(4.28) 

This scalar product is 0 unless

$$2(i - k + l) = n + j - m \tag{4.29}$$

i.e. the number of arrows pointing inside the diagram is equal to the number of arrows pointing outside.

Graphical calculus allows a straightforward computation of the value of the diagram (4.28) by applying Proposition 4.10 to expand one of the three projectors. Let us instead factorize Clebsch-Gordan coefficients via the dual canonical basis of $V_n \otimes V_m$. Here and further we restrict to the case $k \geq l$, the other case being similar.

The transformation from the tensor product basis $\{v^{2l-n}\otimes v^{m-2k}\}_{l,k}$ to the dual canonical basis $\{v^{2l-n}\heartsuit v^{m-2k}\}_{l,k}$ is given by Proposition 1.6(i):

$$v^{2l-n}\otimes v^{m-2k}=\sum_{s\geq 0} q^{-sk}\begin{bmatrix} l \\ s \end{bmatrix} v^{2l-n-2s}\heartsuit v^{m-2k+2s}, \quad k\geq l\geq 0$$

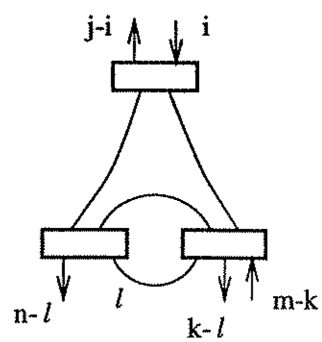
Let $\mathcal{I}(l,k,s)=q^{-sk}\begin{bmatrix} l \\ s \end{bmatrix}$.

The coefficient

$$< v_j^{nm}(v^{2l-n}\heartsuit v^{m-2k}), v^{j-2i} >$$

is equal to the evaluation of the diagram

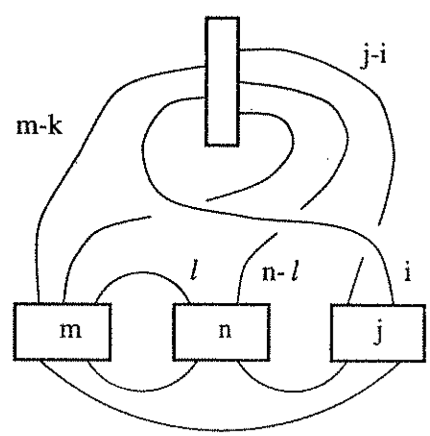
$$A=$$



We want to find $ev(A)$. We impose the condition (4.29), otherwise $ev(A)=0$. Transferring two bottom projectors of A up and closing the diagram, we obtain, by Theorem 2.4,

$$ev(A)=\frac{(-1)^i q^{\frac{(m-k)(i+m-k+2)}{2}}}{[m+i-k+1]}ev(A_1)$$

where the diagram A_1 is

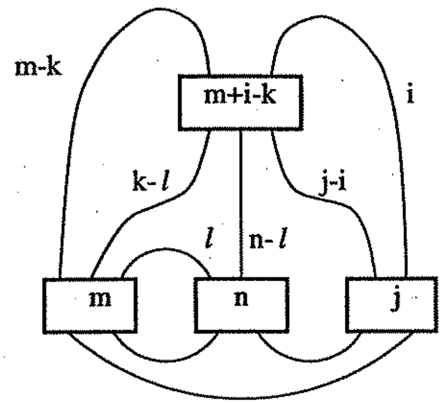


Moving the line marked i over the upper projector and deleting the resulting curl and crossing we get

$$ev(A_1) = (-1)^i q^{\frac{i(m+i-k+2)}{2}} ev(A_2)$$

where

$A_2 =$



Applying Lemma 4.15 twice to the diagram A_2, we reduce A_2 to a theta-curve (see Definition 4.1) and obtain

$$ev(A_2) = \frac{[i]![m-k]![n-l]!}{[i+l-m]![m+i-k]![i+m-j-k]!}\theta(m,n,j)$$

with $\theta(m,n,j)$ given by Theorem 4.11. Combining the formulas together,

$$\begin{aligned} &< v_j^{nm}(v^{2l-n} \heartsuit v^{m-2k}), v^{j-2i} >= ev(A) = \\ (4.30) \qquad &= q^{\mu} \frac{[i]![m-k]![n-l]!}{[i+l-m]![m+i-k+1]![i+m-j-k]!}\theta(m,n,j), \\ &\mu = \frac{1}{2}(m-k)(2i+m-k+2) + \frac{1}{2}i(i+2). \end{aligned}$$

Denote this number by $\mathcal{H}_{nm}^j(l,k,i)$. We now have a factorization of the Clebsch-Gordan coefficient:

PROPOSITION 4.17. *The Clebsch-Gordan coefficient (for $k \geq l$) factorizes through the intermediate basis as follows*

$$\mathcal{C}_{nm}^j(l,k,i) = \sum_{s=0}^{l} \mathcal{I}(l,k,s)\mathcal{H}_{nm}^j(l-s,k,i).$$

CHAPTER V

KAZHDAN-LUSZTIG THEORY AND CANONICAL BASES

5.1. Kazhdan-Lusztig theory in the maximal parabolic case

Here we describe, following Deodhar [D1],[D2], the setting for the relative Kazhdan-Lusztig theory. From the beginning we restrict to the case of the symmetric group $W = \mathbb{S}_n$ and its maximal parabolic subgroup $W_k = \mathbb{S}_k \times \mathbb{S}_{n-k}$. To avoid confusion with the quantum group theory, we will use variable v rather than q. Then v (or q) in Kazhdan-Lusztig theory is related to q in quantum group theory by

$$v = q^{-2} \tag{5.1}$$

Let $\mathcal{H}_n$ be the Hecke algebra over $\mathbb{C}[v^{\frac{1}{2}}, v^{-\frac{1}{2}}]$ of the symmetric group $\mathbb{S}_n$. It has generators $T_1, ..., T_{n-1}$ and relations

$$\begin{aligned} &T_iT_j = T_jT_i \quad |i-j| > 1 \\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \\ &(T_i+1)(T_i - v) = 0 \end{aligned} \tag{5.2}$$

Denote by s_i the transposition $(i, i+1) \in W$. Let $\sigma \in W$ and $\sigma = s_{i_1}...s_{i_r}$ be a reduced expression for σ (i.e. $r = l(\sigma)$.) Then denote $T_\sigma = T_{i_1}...T_{i_r}$.

Let W^k be the set of minimal coset representatives in W/W_k. It's given by

$$W^k = \{\sigma \in W | l(\sigma s) \geq l(\sigma) \quad \forall s \in W_k\}.$$

Deodhar in [D1],[D2] considers two different actions of $\mathcal{H}$ on the space M^k spanned by vectors $m_y^k, y \in W^k$.

Proposition 5.1 (Deodhar). *Let $u = -1$ or $u = v$. Then the following action defines the structure of $\mathcal{H}_n$ module on M^k.*

$$T_i m_y^k = \begin{cases} q m_{s_i y}^k + (q-1) m_y^k & \text{if } s_i y \leq y \\ m_{sy}^k & \text{if } sy \geq y \text{ and } sy \in W^k \\ u m_y^k & \text{if } sy \geq y \text{ and } sy \notin W^k \end{cases}$$

To distinguish the two cases $u = -1$ and $u = v$, in the second case we will denote the module by $\tilde{M}^k$ and the basis vectors by $\tilde{m}_y^k$.

Let $\overline{}$ be the involution (the Kazhdan-Lusztig involution) of $\mathcal{H}$ as a $\mathbb{C}$-algebra given by

$$\overline{v^j} = v^{-j}$$

$$\overline{\sum a_\sigma T_\sigma} = \sum \overline{a_\sigma} T_{\sigma^{-1}}^{-1}$$

In particular, $\overline{T_i} = T_i^{-1}$.

Now define an involution $\overline{}$ on M^k as follows. Let

$$\overline{m_y^k} = \overline{T_y} m_e^k \quad (y \in W^k)$$

and

$$\overline{\sum a_y m_y^k} = \sum \overline{a_y} \overline{m_y^k}$$

It is straightforward to check that $\overline{}$ is an involution:

$$\overline{\overline{m_y^k}} = \overline{\overline{T_y} m_e^k} = \overline{\overline{T_y}}\,\overline{m_e^k} = T_y m_e^k = m_y^k$$

Similarly, define an involution $\overline{}$ on $\tilde{M}^k$ by

$$\overline{\tilde{m}_y^k} = \overline{T_y} \tilde{m}_e^k \quad (y \in W^k)$$

and

$$\overline{\sum a_y \tilde{m}_y^k} = \sum \overline{a_y} \overline{\tilde{m}_y^k}$$

Proposition 5.2(a) (Deodhar). *There exists a unique set of polynomials* $\{P^k_{\tau,s} \in \mathbb{Z}[v] | \tau, s \in W^k, \tau \le s\}$ *satisfying*

(i) $P^k_{s,s} = 1$ *and* $deg_v P^k_{\tau,s} \le (l(s) - l(\tau) - 1)/2$ *if* $\tau < s$,

(ii) Elements $C^k_s = \sum_{\tau \le s} (-1)^{l(s)-l(\tau)} v^{\frac{l(s)}{2}} v^{-l(\tau)} \overline{P^k_{\tau,s}} m^k_\tau$ *are invariant under the involution* $^-$ *of* M^k.

Proposition 5.2(b) (Deodhar). *There exists a unique set of polynomials* $\{\tilde{P}^k_{\tau,s} \in \mathbb{Z}[v] | \tau, s \in W^k, \tau \le s\}$ *satisfying*

(i) $\tilde{P}^k_{s,s} = 1$ *and* $deg_v \tilde{P}^k_{\tau,s} \le (l(s) - l(\tau) - 1)/2$ *if* $\tau < s$,

(ii) Elements $\tilde{C}^k_s = \sum_{\tau \le s} (-1)^{l(s)-l(\tau)} v^{\frac{l(s)}{2}} v^{-l(\tau)} \overline{\tilde{P}^k_{\tau,s}} \tilde{m}^k_\tau$ *are invariant under the involution* $^-$ *of* $\tilde{M}^k$.

The basis $\{C^k_s\}_{s \in W^k}$ (respectively, $\{\tilde{C}^k_s\}_{s \in W^k}$) is called the Kazhdan-Lusztig basis of M^k (respectively, $\tilde{M}^k$).

5.2. Kazhdan-Lusztig basis in M^k and Lusztig's canonical basis

Let us go back to $U_q(\mathfrak{sl}_2)$. We have an intertwiner $\check{R} : V_1^{\otimes 2} \to V_1^{\otimes 2}$. Let us introduce a new intertwiner by scaling $\check{R}$ by $q^{-\frac{3}{2}}$:

$$\tilde{T} = q^{-\frac{3}{2}} \check{R}.$$

Also, let

$$T' = -q^{-2} \tilde{T}^{-1}.$$

Fix an integer k between 0 and n. Consider the subspace $V_1^{\otimes n}[n-2k]$ of vectors of weight $n - 2k$ of the vector space $V_1^{\otimes n}$. Define operators

$$T'_i, \tilde{T}_i : V_1^{\otimes n}[n-2k] \to V_1^{\otimes n}[n-2k]$$

by

$$T_i' = 1^{\otimes(n-i-1)} \otimes T' \otimes 1^{\otimes(i-1)}$$

$$\tilde{T}_i = 1^{\otimes(i-1)} \otimes \tilde{T} \otimes 1^{\otimes(n-i-1)}$$

It's easy to check that operators T_i' and $\tilde{T}_i$ satisfy Hecke algebra relations (5.2) with $v = q^{-2}$.

Recall from Section 5.1 that W^k denotes the set of minimal coset representatives in W/W_k. The set W^k can be identified with the set SQ_k of sequences of $n-k$ ones and k minus ones:

(i) Define an action of W on the set SQ_k in the following way: the trasposition $(i, i+1)$ exchanges $(n-i+1)$-th and $(n-i)$-th elements of a sequence.

(ii) To the identity element $e \in W$, which is also the minimal coset representative of eW_k, associate the sequence of $n-k$ ones followed by k negative ones.

Denote this mapping $W^k \to SQ_k$ by Seq.

Now let us establish relation between the canonical basis in $V_1^{\otimes n}$ and the Kazhdan-Lusztig basis in Deodhar's module M^k. Define the twisted product basis of $V_1^{\otimes n}[n-2k]$ by

$$p[\epsilon_1, ..., \epsilon_n] = (-q^{-1})^{Iv(\epsilon_1,...,\epsilon_n)} v_{\epsilon_1} \otimes ... \otimes v_{\epsilon_n}$$

where $\epsilon_1, ..., \epsilon_n \in \{1, -1\}$ and $Iv(\epsilon_1, ..., \epsilon_n)$ is the number of pairs $(i, j), i < j$ such that $\epsilon_i < \epsilon_j$.

Let Seq be the linear map

$$Seq: \quad M^k \to V_1^{\otimes n}[n-2k]$$

defined on basis vectors by

$$Seq(m_y^k) = p[\epsilon_1, ..., \epsilon_n]$$

where $(\epsilon_1, ..., \epsilon_n)$ is the sequence $Seq(y)$ of ones and negative ones. Notice that we abuse the notations by denoting both the mapping of sets $W^k \to SQ_k$ and the mapping of linear spaces $M^k \to V_1^{\otimes n}[n-2k]$ by Seq.

Denote by $\hat{s}$ the minimal coset representative of an $s \in W$. Let $y \in W^k$. Denote by ω the maximal element of the Weyl group W. Then

$$\omega y = (\widehat{\omega y})\omega(k, n-k)$$

where $\omega(k, n-k)$ is the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n-k & n-k+1 & n-k+2 & \dots & n \\ n-k & n-k-1 & \dots & 1 & n & n-1 & \dots & n-k+1 \end{pmatrix}$$

Therefore,

$$T_{\omega y} = T_{\widehat{\omega y}} T_{\omega(k,n-k)}$$

We have two Hecke algebra modules: M^k, with the action defined in Section 5.1 and $V_1^{\otimes n}[n-2k]$, with Hecke algebra generators acting by $T_i', 1 \leq i \leq n-1$. Denote by $\mathcal{H}_n'$ the Hecke algebra over $\mathbb{C}[q, q^{-1}]$ generated by T_i' with defining relations:

$$T_i'T_j' = T_j'T_i' \quad |i-j| > 1$$

$$T_i'T_{i+1}'T_i' = T_{i+1}'T_i'T_{i+1}'$$

$$(T_i' + 1)(T_i' - q^{-2}) = 0$$

Let Seq_1 be the algebra isomorphism

$$Seq_1 : \mathcal{H}_n \to \mathcal{H}_n'$$

given on generators by

$$Seq_1(v) = q^{-2}$$

$$Seq_1(T_i) = T_i'$$

We have two pairs: $(\mathcal{H}_n, M^k)$ and $(\mathcal{H}_n', V_1^{\otimes n}[n-2k])$, consisting of an algebra and a module over it.

The next proposition follows from the comparison of actions of T_i on M^k and T_i' on $V_1^{\otimes n}[n-2k]$.

PROPOSITION 5.3. *The mappings Seq and Seq_1 define isomorphism of pairs* $(\mathcal{H}_n, M^k)$ *and* $(\mathcal{H}'_n, V_1^{\otimes n}[n-2k])$.

□

To relate Lusztig's involution $\Theta^{(n)}(\sigma^{\otimes n})$ with the involution $\bar{}$ of M^k, Lusztig's involution needs to be twisted by the permutation operator $P^{(n)}$, where $P^{(n)}$ is given by

$$P^{(n)}(x_1 \otimes ... \otimes x_n) = x_n \otimes ... \otimes x_1 \quad (x_i \in V_1)$$

PROPOSITION 5.4. *The following diagram of vector spaces and linear maps is commutative*

$$\begin{array}{ccc} M^k & \xrightarrow{\;\bar{}\;} & M^k \\ \big\downarrow{\scriptstyle Seq} & & \big\downarrow{\scriptstyle Seq} \\ V_1^{\otimes n}[n-2k] & \xrightarrow{P^{(n)}\Theta^{(n)}(\sigma^{\otimes n})P^{(n)}} & V_1^{\otimes n}[n-2k] \end{array}$$

Proof: We have

(i) $T_y m_e^k = m_y^k$ for $y \in W^k$,

(ii) $T_{\omega(k,n-k)} m_e^k = (-1)^{l(\omega(k,n-k))} m_e^k = (-1)^{\frac{n(n-1)}{2} - k(n-k)} m_e^k$,

(iii) $\overline{m_y^k} = \overline{m_y^k} = \overline{T_y} m_e^k$

$= T_{y^{-1}}^{-1} m_e^k = (T_\omega)^{-1} T_\omega T_{y^{-1}}^{-1} m_e^k$

$= (T_\omega)^{-1} T_{\omega y} T_{y^{-1}} T_{y^{-1}}^{-1} m_e^k = (T_\omega)^{-1} T_{\omega y} m_e^k$

$= (T_\omega)^{-1} T_{\hat{\omega} y} T_{\omega(k,n-k)} m_e^k$

We thus derive

$$\overline{m_y^k} = (T_\omega)^{-1} T_{\hat{\omega} y} T_{\omega(k,n-k)} m_e^k =$$

$$(-1)^{l(\omega(k,n-k))} (T_\omega)^{-1} T_{\hat{\omega} y} m_e^k = (-1)^{l(\omega(k,n-k))} (T_\omega)^{-1} m_{\hat{\omega} y}^k$$

To prove the Proposition, it suffices to check that, for any $y \in W^k$,

$$Seq(\overline{m_y^k}) = P^{(n)}\Theta^{(n)}(\sigma^{\otimes n})P^{(n)}(Seq(m_y^k))$$

Define $(T')^{(n)} \in \mathcal{H}$ similarly to $\Theta^{(n)}, P^{(n)}$ and T_ω. Then

$$Seq_1(T_\omega) = (T')^{(n)}$$

and (in the quantum group case)

$$\Theta^{(n)} = (C^{(n)})^{-1} P^{(n)} \check{R}^{(n)}$$

$$C^{(n)} = q^{-\frac{1}{4}[n(n-1)-4k(n-k)]} I$$

$$\check{R}^{(n)} = (-1)^{\frac{n(n-1)}{2}} q^{-\frac{n(n-1)}{4}} ((T')^{(n)})^{-1}$$

We compute

$$
\begin{aligned}
&P^{(n)}\Theta^{(n)}(\sigma^{\otimes n})P^{(n)}(Seq(m_y^k)) \\
&= P^{(n)}\Theta^{(n)}(\sigma^{\otimes n})P^{(n)}[\epsilon_1, ..., \epsilon_n] \\
&= P^{(n)}\Theta^{(n)}(\sigma^{\otimes n})P^{(n)}(-q^{-Iv(\epsilon_1,...,\epsilon_n)})v_{\epsilon_1} \otimes ... \otimes v_{\epsilon_n} \\
&= (-q^{Iv(\epsilon_1,...,\epsilon_n)})P^{(n)}\Theta^{(n)}(\sigma^{\otimes n})v_{\epsilon_n} \otimes ... \otimes v_{\epsilon_1} \\
&= (-q^{Iv(\epsilon_1,...,\epsilon_n)})P^{(n)}(C^{(n)})^{-1}P^{(n)}\check{R}^{(n)}v_{\epsilon_n} \otimes ... \otimes v_{\epsilon_1} \\
&= (-q^{Iv(\epsilon_1,...,\epsilon_n)})(C^{(n)})^{-1}\check{R}^{(n)}v_{\epsilon_n} \otimes ... \otimes v_{\epsilon_1} \\
&= (-q^{Iv(\epsilon_1,...,\epsilon_n)+Iv(\epsilon_n,...,\epsilon_1)})(C^{(n)})^{-1}\check{R}^{(n)}[\epsilon_n, ..., \epsilon_1] \\
&= (-q^{k(n-k)})(C^{(n)})^{-1}\check{R}^{(n)}Seq(m_{\hat{\omega}y}^k) \\
&= (-1)^{k(n-k)}q^{k(n-k)}(C^{(n)})^{-1}(-1)^{\frac{n(n-1)}{2}}q^{-\frac{n(n-1)}{4}}((T')^{(n)})^{-1}Seq(m_{\hat{\omega}y}^k) \\
&= (-1)^{\frac{n(n-1)}{2}+k(n-k)}q^{k(n-k)+\frac{1}{4}[n(n-1)-4k(n-k)]-\frac{n(n-1)}{4}}((T')^{(n)})^{-1}Seq(m_{\hat{\omega}y}^k) \\
&= (-1)^{\frac{n(n-1)}{2}-k(n-k)}((T')^{(n)})^{-1}Seq(m_{\hat{\omega}y}^k) = (-1)^{\frac{n(n-1)}{2}-k(n-k)}Seq_1(T_\omega^{-1})Seq(m_{\hat{\omega}y}^k) \\
&= Seq\left((-1)^{\frac{n(n-1)}{2}-k(n-k)}T_\omega^{-1}m_{\hat{\omega}y}^k\right) = Seq\left((-1)^{l(\omega(k,n-k))}T_\omega^{-1}m_{\hat{\omega}y}^k\right) \\
&= Seq(\overline{m_y^k})
\end{aligned}
$$

This computation finishes the proof of the proposition. □

Having established that the two involutions correspond, we now check that the image under Seq of the Kazhdan-Lusztig basis in M^k is the canonical basis in $V_1^{\otimes n}[n-2k]$ (permuted by $P^{(n)}$).

Applying Seq to an element C_s^k of the Kazhdan-Lusztig basis of M^k, we obtain:

$$
\begin{aligned}
Seq(C_s^k) &= Seq(\sum_{\tau\leq s}(-1)^{l(s)-l(\tau)} v^{\frac{l(s)}{2}} v^{-l(\tau)} \overline{P_{\tau,s}^k} m_\tau^k) \\
&= \sum_{\tau\leq s}(-1)^{l(s)-l(\tau)} q^{-l(s)} q^{2l(\tau)} \overline{P_{\tau,s}^k(q^{-2})} Seq(m_\tau^k) \\
&= \sum_{\tau\leq s}(-1)^{l(s)-l(\tau)} q^{-l(s)} q^{2l(\tau)} \overline{P_{\tau,s}^k(q^{-2})} p[\epsilon_1, ..., \epsilon_n] \\
&(\text{ here } Seq(\tau) = p[\epsilon_1, ..., \epsilon_n]) \\
&= \sum_{\tau\leq s}(-1)^{l(s)-l(\tau)} q^{-l(s)} q^{2l(\tau)} \overline{P_{\tau,s}^k(q^{-2})} (-q^{-Iv(\epsilon_1,...,\epsilon_n)}) v_{\epsilon_1} \otimes ... \otimes v_{\epsilon_n} \\
&(\text{using that } Iv(\epsilon_1, ..., \epsilon_n) = l(\tau)) \\
&= \sum_{\tau\leq s}(-1)^{l(s)} q^{l(\tau)-l(s)} \overline{P_{\tau,s}^k(q^{-2})} v_{\epsilon_1} \otimes ... \otimes v_{\epsilon_n}
\end{aligned}
$$

When $\tau < s$, we have

$$\deg_v\left(v^{\frac{l(s)-l(\tau)}{2}} \overline{P_{\tau,s}^k}\right) \geq \frac{l(s)-l(\tau)}{2} - \frac{(l(s)-l(\tau)-1)}{2} = \frac{1}{2} > 0$$

Therefore, denoting $Seq(s)$ by $(\mu_1, ..., \mu_n)$ and $q^{l(\tau)-l(s)}\overline{P_{\tau,s}^k(q^{-2})}$ by $\psi_{(\mu_1,...,\mu_n)}^{(\epsilon_1,...,\epsilon_n)}$ we obtain

$$(-1)^{l(s)} Seq(C_s^k) = v_{\mu_1} \otimes ... \otimes v_{\mu_n} + \sum_{(\epsilon_1,...,\epsilon_n)>(\mu_1,...,\mu_n)} \psi_{(\mu_1,...,\mu_n)}^{(\epsilon_1,...,\epsilon_n)} v_{\epsilon_1} \otimes ... \otimes v_{\epsilon_n}$$

where

$$\psi_{(\mu_1,...,\mu_n)}^{(\epsilon_1,...,\epsilon_n)} \in q^{-1}\mathbb{Z}[q^{-1}]$$

Besides, as we already know,

$$P^{(n)}\Theta^{(n)}P^{(n)}(\sigma^{\otimes n})Seq(C_s^k) = Seq(C_s^k)$$

Therefore

PROPOSITION 5.5. *The basis* $\{(-1)^{l(s)}Seq(C_s^k) | s \in W^k\}$ *in* $V_1^{\otimes n}[n-2k]$ *coincides with the (permuted by* $P^{(n)}$*) canonical basis in* $V_1^{\otimes n}[n-2k]$.

5.3. Kazhdan-Lusztig basis in $\tilde{M}^k$ and dual canonical basis

Let us identify W^k with the set SQ_k of sequences of $n-k$ ones and k minus ones in a way different from the correspondence Seq of the previous section.

(i) Define an action of W on the set SQ_k as follows: the trasposition $(i, i+1)$ exchanges i-th and $(i+1)$-st elements of a sequence.

(ii) To the identity element $e \in W$, which is also the minimal coset representative of eW_k, associate the sequence of k negative ones followed by $n-k$ ones.

Denote this mapping $W^k \to SQ_k$ by $\tilde{Seq}$. Now define another basis in $V_1^{\otimes n}[n-2k]$ by

$$\tilde{p}[\epsilon_1, ..., \epsilon_n] = q^{Iv(\epsilon_1,...,\epsilon_n)-k(n-k)} v_{\epsilon_1} \otimes ... \otimes v_{\epsilon_n}$$

Also denote by $\tilde{Seq}$ the linear map

$$\tilde{Seq}: \quad \tilde{M}^k \to V_1^{\otimes n}[n-2k]$$

given on basis vectors by

$$\tilde{Seq}(\tilde{m}_y^s) = \tilde{p}[\epsilon_1, ..., \epsilon_n]$$

where $(\epsilon_1, ..., \epsilon_n)$ is the sequence $\tilde{Seq}(s)$ of ones and negative ones.

Note that the Hecke algebra $\mathcal{H}_n$ acts on $V_1^{\otimes n}[n-2k]$ via operators $\tilde{T}_i$. Denote by $\tilde{Seq}_1$ the map of Hecke algebras (we have two copies of the Hecke algebra: one acting on $\tilde{M}^k$, another on $V_1^{\otimes n}[n-2k]$):

$$v \to q^{-2}, \quad T_i \to \tilde{T}_i \quad (1 \le i \le n-1).$$

Similar to Proposition 5.3, we have

PROPOSITION 5.6. *Mappings $\tilde{Seq}$ and $\tilde{Seq}_1$ define isomorphism of pairs $(\mathcal{H}_n, \tilde{M}^k)$ and $(\mathcal{H}_n, V_1^{\otimes n}[n-2k])$.*

□

We next prove

PROPOSITION 5.7. *The following diagram of vector spaces and linear maps is commutative:*

$$\begin{array}{ccc} \tilde{M}^k & \xrightarrow{\ \ -\ \ } & \tilde{M}^k \\ \Big\downarrow \tilde{Seq} & & \Big\downarrow \tilde{Seq} \\ V_1^{\otimes n}[n-2k] & \xrightarrow{\overline{\Theta^{(n)}}(\sigma^{\otimes n})} & V_1^{\otimes n}[n-2k] \end{array}$$

Proof: We mimic the proof of Proposition 5.4 substituting in certain places powers of q for powers of negative one. Specifically, we get

$$T_{\omega(k,n-k)}\tilde{m}_e^k = v^{l(\omega(k,n-k))}\tilde{m}_e^k,$$

$$\overline{\tilde{m}_y^k} = v^{l(\omega(k,n-k))}(T_\omega)^{-1}T_{\hat{\omega}y}\tilde{m}_{\hat{\omega}y}^k$$

From these identities we deduce

$$\overline{\Theta^{(n)}}(\sigma^{\otimes n})\tilde{p}[\epsilon_1, ..., \epsilon_n] = q^{\frac{n(n-1)}{2}-k(n-k)}(\tilde{T}^{(n)})^{-1}\tilde{p}[\epsilon_n, ..., \epsilon_1]$$

which easily implies the claim of the proposition.

□

PROPOSITION 5.8. *The Kazhdan-Lusztig basis $\{\tilde{C}_s^k\}_{s\in W^k}$ of $\tilde{M}^k$ maps under $\tilde{Seq}$ to the dual canonical basis of $V_1^{\otimes n}[n-2k]$.*

Proof is completely parallel to the proof of Proposition 5.5 and is omitted. We only note that due to the scaling difference between

$$p[\epsilon_1, ..., \epsilon_n] = (-q^{-1})^{Iv(\epsilon_1,...,\epsilon_n)}v_{\epsilon_1}\otimes ... \otimes v_{\epsilon_n}$$

and

$$\tilde{p}[\epsilon_1, ..., \epsilon_n] = q^{Iv(\epsilon_1,...,\epsilon_n)-k(n-k)}v_{\epsilon_1}\otimes ... \otimes v_{\epsilon_n}$$

the off-diagonal coefficients of the decomposition of $\tilde{C}_s^k$ in the product basis of $V_1^{\otimes n}$ will lie in $q\mathbb{Z}[q]$ rather than in $q^{-1}\mathbb{Z}[q^{-1}]$, and that the powers of negative one present in the identification of $\{(-1)^{l(s)}Seq(C_s^k)\}_s$ with the canonical basis disappears in this case.

□

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