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REAL $K(\pi, 1)$ ARRANGEMENTS FROM FINITE ROOT SYSTEMS

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ABSTRACT. Consider the arrangement of codimension two subspaces of an n-dimensional Euclidean space $\mathbb{R}^n = \{(x_1, ..., x_n) | x_i \in \mathbb{R}\}$, that consists of triple diagonals $x_i = x_j = x_k$ for all $1 \le i < j < k \le n$. We answer positively A.Björner's question whether the complement of this arrangement is a $K(\pi, 1)$ space. We construct some other $K(\pi, 1)$ arrangements and show that they come naturally from finite root systems.

0. Introduction

Take a complex n-dimensional space $\mathbb{C}^n$ and delete all diagonals $z_i = z_j$. It is well-known that what remains is a $K(\pi, 1)$ space with the fundamental group isomorphic to the pure braid group ([FN]).

This paper originated from an idea for finding a real counterpart of this construction. Starting from a real n-dimensional space $\mathbb{R}^n$ remove either all real codimension two subspaces $x_i = x_j = x_k$ or all real codimension two subspaces $x_i = x_j, x_k = x_p$ for distinct integers i, j, k, p. Denote the complements by $\mathbf{X}_n$ and $\mathbf{Y}_n$ respectively. We prove here that these are $K(\pi, 1)$'s.

The question of whether $\mathbf{X}_n$ is a $K(\pi, 1)$ was posed by Anders Bjorner ([B1], page 362). Bjorner denoted this space $M_{\mathcal{A}_{n,3}}$.

The pure braid group is the kernel of a homomorphism from the braid group to the symmetric group. Mirroring this, we explicitly realize the fundamental groups of $\mathbf{X}_n$ and $\mathbf{Y}_n$ as kernels of homomorphisms of certain Coxeter groups to the symmetric group (the difference is that the braid group is not a Coxeter group.) We call these Coxeter groups the twin and the triplet groups respectively and their elements twins and triplets. Intuitively, twin and triplet groups are real forms of the braid group.

The classical result that $\mathbb{C}^n$ without diagonals is a $K(\pi, 1)$ admits the following well-known generalization. Deligne [D] proved that the complexification of a real simplicial hyperplane arrangement is a $K(\pi, 1)$ arrangement. In particular, the complexification of the reflection arrangement

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of a finite root system is a $K(\pi,1)$ complex arrangement. This theorem and our results lead us to state a conjecture that certain real codimension two arrangements naturally associated to finite root systems are $K(\pi,1)$ arrangements (see §4).

Here is the plan of the paper. In §1 we describe a class of real codimension two arrangements and prove that all arrangements in this class are $K(\pi,1)$ arrangements. As a special case, $\mathbf{X}_n$ is a $K(\pi,1)$ space. In §2 we prove that $\mathbf{Y}_n$ is homotopy equivalent to a bouquet of circles, and, consequently, a $K(\pi,1)$ space. In §3 we describe fundamental groups of $\mathbf{X}_n, \mathbf{Y}_n$ and several other similar spaces. In §4 we establish a relation between our arrangements and finite root systems and propose a conjecture that would clarify this relation.

Braids can be described combinatorially via their projections to a plane. Similar realizations of twins and triples are given in §3.

The ideas of the paper have the following interesting development. Recall that braids relate to links via the closure operation. It turns out that twins are related to doodles—collections of closed curves on the two-sphere without triple intersections ([FT],[K1]). As the fundamental group of the link complement is an invariant of a link, to a doodle there is associated a "fundamental" group ([K1]). The twin group on n arcs acts in a special way on the free group of rank $n+1$ so that the "fundamental" group of the closure of a twin is isomorphic to the quotient of the free group by the automorphism associated to this twin ([K2]). These and other features of doodles mirror those of links.

We would like to remark that Vassiliev's ornaments [V] are equivalence classes of doodles relative to triple intersections of one or two different components (but not three different components).

Triplets are related to objects that we call noodles: fix a codimension 1 foliation (with singular points) on the two-sphere. Noodles are collections of closed curves on the two-sphere such that no two intersection points belong to the same leaf, there are no quadruple intersections and no intersection point can occupy a singular point of the foliation. Noodles admit a group-valued invariant which is of independent interest.

I am grateful to Anders Bjorner for stimulating discussions and encouragement and my advisor Igor Frenkel for his support.

1. Examples of real $K(\pi,1)$ arrangements

Recall that an Eilenberg-MacLane space (or a $K(\pi,n)$ space) is a connected cell complex with all homotopy groups except the n-th homotopy group being trivial and the n-th homotopy group isomorphic to π.

Let $V_n(3)$ be the set of triples (i, j, k) of pairwise different integers between 1 and n. Let $V_n(4)$ be the set of quadruples (i, j, k, l) of pairwise different integers between 1 and n.

For a subset $S \subset V_n(4)$ define the space $\mathbf{X}_{n,S}$:

$$\mathbf{X}_{n,S} = \mathbb{R}^n \setminus \{(x_1, ..., x_n) | x_i = x_j = x_k \text{ for } 1 \leq i < j < k \leq l,$$
$$x_i = x_j, \quad x_k = x_l \text{ for } (i, j, k, l) \in S\}.$$

$\mathbf{X}_{n,S}$ is obtained from $\mathbb{R}^n$ by deleting all triple diagonals $x_i = x_j = x_k$ and deleting the real codimension 2 hyperplanes $x_i = x_j, x_k = x_l$ with $(i, j, k, l) \in S$.

Theorem 1.1. *For any $n > 1$ and $S \subset V_n(4)$ the space $\mathbf{X}_{n,S}$ is a $K(\pi, 1)$ space.*

Proof. Fix n and $S \subset V_n(4)$. We must show that for $p > 1$ any continuous mapping of the p-dimensional sphere $\mathbb{S}^p$ to $\mathbf{X}_{n,S}$ is contractible.

To a number p, $p > 1$ and a mapping $\psi : \mathbb{S}^p \to \mathbf{X}_{n,S}$ we associate the following data: Take a trivial line bundle $\mathbb{S}^p \times \mathbb{R}$ over $\mathbb{S}^p$. Let $\psi_1, ...\psi_n$ be the compositions of ψ with the projections pr_i of $\mathbf{X}_{n,S} \subset \mathbb{R}^n$ onto the coordinates of $\mathbb{R}^n$:

$$pr_i : (x_1, ..., x_n) \to x_i, \quad \psi_i \stackrel{\text{def}}{=} pr_i \circ \psi, \quad \psi_i : \mathbb{S}^p \to \mathbb{R}.$$

In the trivial bundle $\mathbb{S}^p \times \mathbb{R}$ we get n continuous sections

$$\phi_i : \mathbb{S}^p \to \mathbb{S}^p \times \mathbb{R}, \quad \phi_i(y) = (y, \psi_i(y)) \text{ for } y \in \mathbb{S}^p.$$

The sections $\phi_1(\mathbb{S}^p), ..., \phi_n(\mathbb{S}^p)$ are n manifolds homeomorphic to $\mathbb{S}^p$ inside $\mathbb{S}^p \times \mathbb{R}$.

The condition that ψ is a mapping into $\mathbf{X}_{n,S}$ is equivalent to the following two conditions on $\phi_1(\mathbb{S}^p), ..., \phi_n(\mathbb{S}^p)$:
(i) Manifolds $\phi_1(\mathbb{S}^p), ..., \phi_n(\mathbb{S}^p)$ have only double points of intersection,
(ii) If for a point $y \in \mathbb{S}^p$ and a quadruple $(i, j, k, l) \in S$ we have $\phi_i(y) = \phi_j(y)$ then $\phi_k(y) \neq \phi_l(y)$.

Also, ψ is a C^∞-mapping iff the manifolds $\phi_1(\mathbb{S}^p), ..., \phi_n(\mathbb{S}^p)$ are C^∞-submanifolds of $\mathbb{S}^p \times \mathbb{R}$.

Pick an integer p greater than 1 and a continuous mapping $\psi : \mathbb{S}^p \to \mathbf{X}_{n,S}$. We will prove that ψ is homotopic to a mapping into a point.

First, by a small deformation we can make ψ a C^∞ mapping. Thus, w.l.o.g. we suppose that ψ is C^∞. Next, deforming ψ a little if necessary (and keeping it C^∞ during deformation), we suppose that for any i, j the intersection of $\psi(\mathbb{S}^p)$ and the diagonal $x_i = x_j$ of $\mathbb{R}^n$ is a smooth manifold (not necessarily connected) of dimension $p-1$.

To ψ there is associated (as described above) n smooth sections $\phi_1(\mathbb{S}^p), ..., \phi_n(\mathbb{S}^p)$ of the trivial bundle $\mathbb{S}^p \times \mathbb{R}$. These n sections can have only double intersections and the intersections are smooth manifolds of dimension $p-1$.

Denote by $P_1, ..., P_u$ the connected components of these intersections:

$$P_1 \cup P_2 \cup ... \cup P_u = \cup_{i \neq j}(\phi_i(\mathbb{S}^p) \cap \phi_j(\mathbb{S}^p)).$$

Manifolds $P_1, ..., P_u$ cut the p-dimensional spheres $\phi_1(\mathbb{S}^p), ..., \phi_n(\mathbb{S}^p)$ into the union of connected manifolds $L_1, ..., L_v$ with boundary. Each $L_1, ..., L_v$ has dimension p and the boundary of each $L_1, ..., L_v$ is a disjoint union of some of $P_1, ..., P_u$. Observe that under the projection of $\mathbb{S}^p \times \mathbb{R}$ onto the base $\mathbb{S}^p$ each of $L_1, ..., L_v$ projects diffeomorphically onto a domain of $\mathbb{S}^p$.

Because $p > 1$, for any $i, 1 \leq i \leq n$, among those $L_1, ..., L_v$ that are submanifolds of $\phi_i(\mathbb{S}^p)$ there is at least one with connected boundary. Take a Riemannian metric on $\mathbb{S}^p$.

There exists a number w between 1 *and v such that the boundary of L_w is connected and the projection of L_w on $\mathbb{S}^p$ (onto the base of the bundle $\mathbb{S}^p \times \mathbb{R}$) has the minimal volume among the volumes of all projections $p : L_l \to \mathbb{S}^p, \quad l = 1, ..., v$ with the boundary of L_l connected.*

Now we look at L_w and its boundary ∂L_w. We can deform n sections $\phi_1(\mathbb{S}^p), ..., \phi_n(\mathbb{S}^p)$ of $\mathbb{S}^p \times \mathbb{R}$ so as to destroy the set ∂L_w of double points. The deformation is schematically depicted on Figure 1.

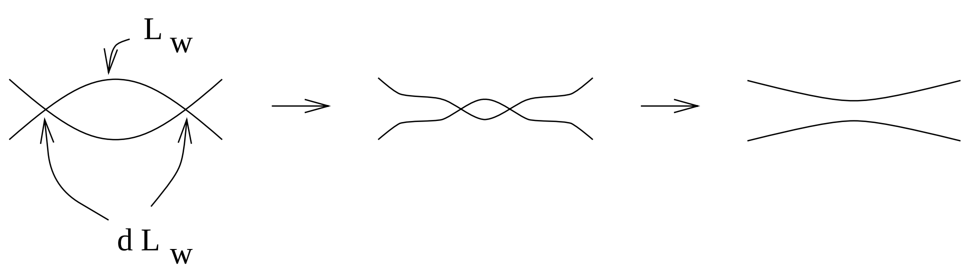

Figure 1

During the homotopy only two of the sections–those that contain ∂L_w are deformed.

Because of the way we chose w, at any time during the homotopy conditions (i) and (ii) are preserved. Thus, to this homotopy of sections there is associated a homotopy of $\mathbb{S}^p$ inside $\mathbf{X}_{n,S}$. After this homotopy the number u of connected components of $\cup_{i \neq j}(\phi_i(\mathbb{S}^p) \cap \phi_j(\mathbb{S}^p))$ diminishes by 1.

Doing this type of homotopies (each time finding w satisfying conditions as above) over and over we get to a mapping $\hat{\psi} : \mathbb{S}^p \to \mathbf{X}_{n,S}$ such that $\hat{\psi}(\mathbb{S}^p)$ does not intersect any of the diagonals $x_i = x_j$. Then the sphere $\hat{\psi}(\mathbb{S}^p)$ is contractible inside $\mathbf{X}_{n,S}$. Mappings ψ and $\hat{\psi}$ are homotopic. Thus, the sphere $\psi(\mathbb{S}^p)$ is contractible inside $\mathbf{X}_{n,S}$ and $\mathbf{X}_{n,S}$ is an Eilenberg-Maclane space. □

Denote by $\mathbf{X}_n$ the space $\mathbf{X}_{n,S}$ with S being the empty set. $\mathbf{X}_n$ is the Euclidean space $\mathbb{R}^n$ without triple diagonals $x_i = x_j = x_k$.

Corollary 1.2. $\mathbf{X}_n$ *is a* $K(\pi,1)$ *space.*

For integers n, r such that $n > 1, r \geq 0$ denote by $V_{n,r}(4)$ the set of quadruples (i,j,k,l) of integers such that

$$i \neq j, k \neq l, \quad 1 \leq i,j \leq n, \quad 1 \leq k,l \leq r.$$

Pick an increasing sequence of length r of real numbers $a_1 < a_2 < ... < a_r$ and two sets S and S' where $S \subset V_n(4), S' \subset V_{n,r}(4)$. Remove from the space $\mathbf{X}_{n,S}$ all points that belong to the union of the following $(n-2)$-dimensional subspaces of $\mathbb{R}^n$:

$$a_i = x_j = x_k \text{ for } 1 \leq i \leq r, 1 \leq j < k \leq n$$
$$x_i = a_k, x_j = a_l \text{ for } (i,j,k,l) \in S'.$$

Denote the resulting space by $\mathbf{X}_{n,S,S'}(a_1,...,a_r)$. Observe that for any two sequences $a_1 < ... < a_r$ and $b_1 < ... < b_r$ the manifolds $\mathbf{X}_{n,S,S'}(a_1,...,a_r)$ and $\mathbf{X}_{n,S,S'}(b_1,...,b_r)$ are diffeomorphic.

Theorem 1.3. *For any* $n > 1, r \geq 0, S \subset V_n(4), S' \subset V_{n,r}(4)$ *and a sequence* $a_1 < ... < a_r$ *the space* $\mathbf{X}_{n,S,S'}(a_1,...,a_r)$ *is a* $K(\pi,1)$*-space.*

Proof is almost identical with the proof of theorem 1.1. To a mapping $\psi : \mathbb{S}^p \to \mathbf{X}_{n,S,S'}(a_1,...,a_r)$ we associate n sections $\phi_1,...,\phi_n$ as before and, in addition, r constant sections $\zeta_1,...,\zeta_r$:

$$\zeta_i(y) = (y, a_i), \quad y \in \mathbb{S}^p, 1 \leq i \leq r.$$

We perturb ψ so as to make it C^∞. The $n+r$ sections defined above have only double intersections and, as in the proof of theorem 1.1, we remove connected components of the set of double intersections one by one. We omit technicalities. □

Theorem 1.3 specializes to theorem 1.1 when $r = 0$.

Recall that $V_n(3)$ and $V_n(4)$ are the sets of triples and quadruples of pairwise different integers between 1 and n. Let M_n be disjoint union of $V_n(4)$ and $V_n(3)$.

Pick $n > 0, r \geq 0, S \subset M_n, S' \subset V_{n,r}(4)$ and an increasing sequence $0 < a_1 < ... < a_r$. Consider the following arrangements of codimension 2 subspaces of $\mathbb{R}^n$:

$$\begin{aligned}
&x_i = \epsilon x_j = \delta x_k, \quad \epsilon, \delta \in \{+1, -1\}, \quad 1 \leq i < j < k \leq n \\
&x_i = x_j = 0, \quad 1 \leq i < j \leq n \\
&x_i = \epsilon x_j, \quad x_k = \delta x_l \text{ for } (i,j,k,l) \in S, \quad \epsilon, \delta \in \{+1, -1\} \\
&x_i = 0, \quad x_j = \epsilon x_k \text{ for } (i,j,k) \in S, \quad \epsilon \in \{+1, -1\} \\
&a_i = \epsilon x_j = \delta x_k, \quad 1 \leq i \leq r, 1 \leq j < k \leq n, \quad \epsilon, \delta \in \{+1, -1\} \\
&x_i = \epsilon a_k, x_j = \delta a_l \text{ for } (i,j,k,l) \in S', \quad \epsilon, \delta \in \{+1, -1\}.
\end{aligned}$$

Denote the complement of this arrangement by $\mathbf{Q}_{n,S,S'}(a_1, ..., a_r)$. When $r = 0$ and, in consequence, the set S' is empty, we also denote the complement by $\mathbf{Q}_{n,S}$.

Theorem 1.4. *For any $n, r, S, S', a_1, ..., a_r$ as above $\mathbf{Q}_{n,S,S'}(a_1, ..., a_r)$ is a $K(\pi, 1)$-space.*

Proof is analogous to the proof of theorem 1.1. The difference is that now we manipulate $2n + 2r$ sections $\phi_1^{\pm}, ..., \phi_n^{\pm}, \zeta_1^{\pm}, ..., \zeta_r^{\pm}$ of the bundle $\mathbb{S}^p \times \mathbb{R}$:

$$\begin{aligned}
&\phi_i^{\pm}(y) = (y, \pm\psi_i(y)), \quad y \in \mathbb{S}^p, \quad 1 \leq i \leq n \\
&\zeta_i^{\pm}(y) = (y, \pm a_i), \quad y \in \mathbb{S}^p, \quad 1 \leq i \leq r
\end{aligned}$$

where $\psi_i = pr_i \circ \psi$ as in the proof of theorem 1.1. Again, we omit the details. $\square$

Corollary 1.5. *For any $S \subset M_n$ the space $\mathbf{Q}_{n,S}$ is a $K(\pi, 1)$-space.* $\square$

Among arrangements $\mathbf{Q}_{n,S,S'}(a_1, ..., a_r)$ the most interesting ones are those with $r = 0$ and $S = V_3(n)$ or $V_4(n)$ or M_n (see §4), as they have the biggest symmetry groups.

Fix n and $S \subset V_n(4)$. Take the n-dimensional torus $(\mathbb{S}^1)^{\times n}$. We can specify a point of $(\mathbb{S}^1)^{\times n}$ by its coordinates $(y_1, ..., y_n), y_i \in \mathbb{S}^1$.

Consider the closed subset formed by $(n-2)$-dimensional subtori of $(\mathbb{S}^1)^{\times n}$ consisting of points with coordinates

$$\begin{aligned}
&y_i = y_j = y_k, \quad 1 \leq i < j < k \leq n \\
&y_i = y_j, y_k = y_l \quad (i,j,k,l) \in S.
\end{aligned} \tag{1.2}$$

Theorem 1.6. *For any n and any $S \subset M_n$ the complement of (1.2) in the n-dimensional torus $(\mathbb{S}^1)^{\times n}$ is a $K(\pi, 1)$-space.*

Proof is obtained from the proof of theorem 1.1 by changing $\mathbb{R}$ to $\mathbb{S}^1$ everywhere. $\square$

2. $K(\pi, 1)$ arrangements with a free fundamental group

Let A be a real hyperplane arrangement in $\mathbb{R}^n$. Denote by $A_{(2)}$ the arrangement consisting of intersections of all pairs of hyperplanes from A.

Hyperplanes from A cut $\mathbb{R}^n$ into the union of cells (each cell homeomorphic to the interior of a k-dimensional ball, $0 \leq k \leq n$). This decomposition, restricted to $\mathbb{R}^n \setminus A_{(2)}$, partitions it into a union of n and $(n-1)$-dimensional open cells. The intersection of the closures (inside $\mathbb{R}^n \setminus A_{(2)}$) of any subset of these cells is either empty or contractible. By the nerve theorem [Bo] the space $\mathbb{R}^n \setminus A_{(2)}$ has the homotopy type of a wedge of circles.

Taking arrangement A to be $x_i = x_j, \quad 1 \leq i < j \leq n$ or

$$x_i = \epsilon x_j \quad \epsilon \in \{1, -1\}, 1 \leq i < j \leq n, \quad x_i = 0 \quad 1 \leq i \leq n$$

we conclude that the spaces $\mathbf{X}_{n,V_n(4)}$ and $\mathbf{Q}_{n,M_n}$ have the homotopy type of a wedge of circles.

Denote by $\mathbf{Y}_n$ the space

$$\mathbb{R}^n \setminus \{(x_1, ..., x_n) | x_i = x_j, x_k = x_l, \quad i, j, k, l \text{ are pairwise different}\}$$

$\mathbf{Y}_n$ is the Euclidean space $\mathbb{R}^n$ without the codimension two subspaces $x_i = x_j, x_k = x_l$ for all quadruples (i, j, k, l) of pairwise different integers between 1 and n. For symmetry reasons that will become apparent in proposition 3.2 we now prove

Theorem 2.1. *$\mathbf{Y}_n$ is homotopy equivalent to a bouquet of circles.*

Proof. Hyperplane arrangement $x_i = x_j, 1 \leq i < j \leq n$ decomposes $\mathbb{R}^n$ into cells

$$x_{\sigma(1)} < ... < x_{\sigma(j_1)} = ... = x_{\sigma(k_1)} < ... = x_{\sigma(k_s)} < ... < x_{\sigma(n)}$$

where $\sigma \in \mathbf{S}_n$. This decomposition, restricted to $\mathbf{Y}_n$, partitions $\mathbf{Y}_n$ into the union of $n, (n-1), (n-2)$-cells as follows

$$\begin{aligned}
&n\text{-cells:} x_{\sigma(1)} < ... < x_{\sigma(n)}, \sigma \in \mathbf{S}_n,\\
&(n-1)\text{-cells:} x_{\sigma(1)} < ... < x_{\sigma(t)} = x_{\sigma(t+1)} < ... < x_{\sigma(n)},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \sigma \in \mathbf{S}_n, 1 \leq t \leq n-1,\\
&(n-2)\text{-cells: } x_{\sigma(1)} < ... < x_{\sigma(t)} = x_{\sigma(t+1)} = x_{\sigma(t+2)} < ... < x_{\sigma(n)},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \sigma \in \mathbf{S}_n, 1 \leq t \leq n-2.
\end{aligned}$$

Denote by SC_n the set of these cells. Define a partial order on SC_n such that for $c_1, c_2 \in SC_n$, $c_1 \leq c_2$ iff the closure of c_2 contains c_1.

It is easy to verify that the intersection of the closures of any subset of cells from SC_n is either empty or contractible. By the nerve theorem, the space $\mathbf{Y}_n$ has the homotopy type of the order complex $K(SC_n)$ of the partially ordered set SC_n. The vertices of the order complex $K(SC_n)$ has vertices–elements of set SC_n and simplices are formed by linearly ordered subsets of SC_n (see [O]).

The 2-dimensional CW-complex Γ_0 we describe below is easily seen to be homeomorphic to the order complex $K(SC_n)$.

Γ_0 has $n!$ vertices denoted $(i_1, ..., i_n)$, where $i_1, ..., i_n$ is a permutation of $1, 2, ..., n$.

Γ_0 has $\frac{n!(n-1)}{2}$ edges denoted $(i_1, ..., i_n; t)$ where $i_1, ..., i_n$ is a permutation of $1, 2, ..., n$; t is an integer between 1 and $n-1$ and $i_t < i_{t+1}$. The edge $(i_1, ..., i_n; t)$ connects vertices $(i_1, ..., i_n)$ and $(i_1, ..., i_{t-1}, i_{t+1}, i_t, i_{t+2}, ..., i_n)$.

Γ_0 has $\frac{n!(n-2)}{6}$ two-dimensional cells denoted $(i_1, ..., i_n; t)'$, where $i_1, ..., i_n$ is a permutation of $1, 2, ..., n$; t is an integer between 1 and $n-2$ and $i_t < i_{t+1} < i_{t+2}$. The 2-cell $(i_1, ..., i_n; t)'$ is a hexagon whose 6 edges are glued to the 1-cells

$$\begin{aligned}
&(i_1, ..., i_n; t),\\
&(i_1, ..., i_{t-1}, i_{t+2}, i_t, i_{t+1}, i_{t+3}, ..., i_n; t+1),\\
&(i_1, ..., i_n; t+1),\\
&(i_1, ..., i_{t-1}, i_{t+1}, i_{t+2}, i_t, i_{t+3}, ..., i_n; t),\\
&(i_1, ..., i_{t-1}, i_{t+1}, i_t, i_{t+2}, ..., i_n; t+1),\\
&(i_1, ..., i_t, i_{t+2}, i_{t+1}, i_{t+3}, ..., i_n; t)
\end{aligned}$$

of Γ_0.

We will prove that Γ_0 can be retracted onto a part of its 1-skeleton. First, edges of Γ_0 of the form $(i_1, ..., i_n; 1)$, where $i_1 < i_2 < i_3$, belong to the closure of only 1 two-dimensional cell: cell $(i_1, ..., i_n; 1)'$.

Denote by β_1 the union of open 1- and 2-cells $(i_1, ..., i_n; 1)$ and $(i_1, ..., i_n; 1)', i_1 < i_2 < i_3$. Denote by Γ_1 the complement of β_1 in Γ_0.

More generally, define set β_t, where $1 \leq t \leq n-2$ to be the union of 1-cells

$$(i_1, ..., i_n; t), i_t < i_{t+1} < i_{t+2} \tag{2.1}$$

and 2-cells

$$(i_1, ..., i_n; t)', i_t < i_{t+1} < i_{t+2}. \tag{2.2}$$

Define inductively Γ_t as the complement of β_t in Γ_{t-1}.

Any 1-cell (2.1) of β_t belongs to the closure of exactly one 2-cell (cell (2.2)) of Γ_{t-1} and this 2-cell also belongs to β_t. Hence, Γ_{t-1} and $\Gamma_t = \Gamma_{t-1} \setminus \beta_t$ are homotopy equivalent. The explicit homotopy is the identity on Γ_t and contracts cells (2.1) and (2.2) onto $\partial(i_1, ..., i_n; t)' \setminus (i_1, ..., i_n; t)$.

Therefore, Γ_{n-2} is homotopy equivalent to Γ_0. Note that Γ_{n-2} is one-dimensional. Hence, CW-complex Γ_0 has the homotopy type of a one-dimensional CW-complex. That implies theorem 2.1 because $\mathbf{Y}_n$ is homotopy equivalent to Γ_0. $\square$

3. Fundamental groups

Here we realize fundamental groups of $\mathbf{X}_n, \mathbf{Q}_n$, $\mathbf{Q}_{n,V_n(4)}$, $\mathbf{Q}_{n,V_n(3)}$ as normal subgroups of finite index of certain groups that admit simple description by generators and relations. Namely, in each case we have a Coxeter group G, a Weil group W of the root system A_n or B_n and an epimorphism $G \to W$. The fundamental group of the space is isomorphic to the kernel of this homomorphism.

Take a Euclidean plane $\mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$ and two parallel lines $y = 1$ and $y = 0$. Fix $n \in \{1, 2, ...\}$. Choose n points on the line $y = 1$, say, points $(1, 1), (2, 1), ..., (n, 1)$. Take n points $(1, 0), (2, 0), ..., (n, 0)$ on the line $y = 0$.

Denote by $\mathcal{F}_n$ the topological space of configurations of n continuous arcs in the strip $\mathbb{R} \times [0, 1]$ between the lines $y = 0$ and $y = 1$ such that
(i) arcs connect points $(1, 1), ..., (n, 1)$ with $(1, 0), ..., (n, 0)$ (in some order),
(ii) arcs descent monotonically and never go upward (i.e. each arc intersects each line $y = c$ $(0 \leq c \leq 1)$ in one point),
(iii) no three arcs have a common point.

An example of such configuration is depicted on figure 2.

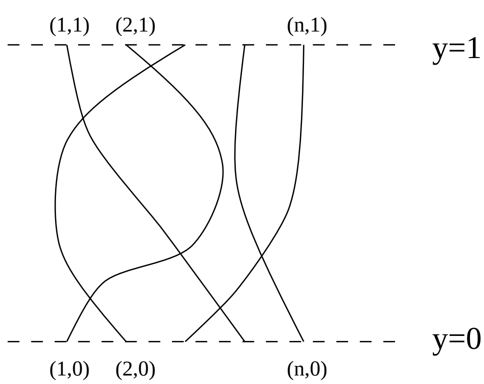

Figure 2

Two elements $a, b \in \mathcal{F}_n$ are multiplied by putting one on top of the other and squeezing the interval $[0, 2]$ to $[0, 1]$.

Definition 3.1. *A twin on n arcs is a connected component of the space $\mathcal{F}_n$.*

The product in $\mathcal{F}_n$ descends to an associative product on the set of twins on n arcs. Each twin can be written as a product of twins p_i, depicted on figure 3. The twin p_i has a configuration with only one double point.

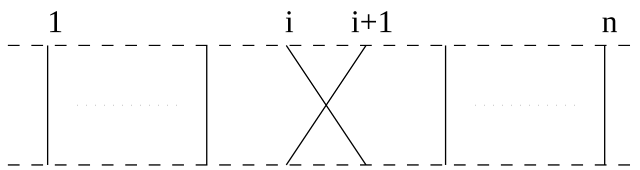

Figure 3

$p_i^2 = 1$ because the two configurations on figure 4 belong to the same connected component of $\mathcal{F}_n$.

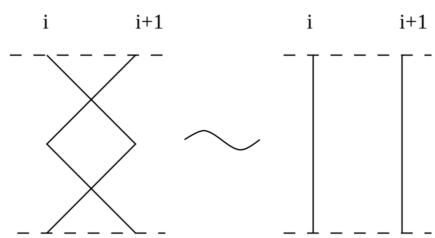

Figure 4

Therefore, each twin has the inverse and twins constitute a group. We call it *the twin group on n arcs* and denote TW_n. The twin group TW_n admits a presentation by generators $p_i, 1 \leq i \leq n-1$ and defining relations

$$p_i^2 = 1, \quad 1 \leq i \leq n-1$$
$$p_i p_j = p_j p_i, \quad |i-j| > 1, \quad 1 \leq i, j \leq n-1.$$

The Yang-Baxter relation $p_i p_{i+1} p_i = p_{i+1} p_i p_{i+1}$ is missing because we do not allow triple intersections.

There is a natural homomorphism of the twin group to the symmetric group $\mathbf{S}_n$ given by sending p_i to the transposition $(i, i+1) \in \mathbf{S}_n$. The kernel of this homomorphism is called *the pure twin group on n arcs.*

Proposition 3.1. *The fundamental group of the space $\mathbf{X}_n$ (the Euclidean space $\mathbb{R}^n$ without triple diagonals $x_i = x_j = x_k$) is isomorphic to the pure twin group on n arcs.*

Indeed, paths in $\mathbf{X}_n$ starting in the point $(1,2,...,n)$ and terminating in $(\sigma(1),...,\sigma(n))$ (where $\sigma \in \mathbf{S}_n$) are in one-to-one correspondence with configurations of n arcs satisfying conditions described above (i.e. with points of $\mathcal{F}_n$). Next, the elements of the fundamental group of $\mathbf{X}_n$ (homotopy classes of closed paths in $\mathbf{X}_n$ with the basepoint $(1,2,...,n)$) are in one-to-one correspondence with the pure twins on n arcs. The multiplication of twins coincides with the multiplication in $\pi_1(\mathbf{X}_n)$. Proposition follows.□

Remark: A special case of the results of Bjorner and Welker [BW] says that $H^i(\mathbf{X}_n,\mathbb{Z})$ are free, $H^i(\mathbf{X}_n,\mathbb{Z}) \neq 0$ iff $0 \leq i \leq \frac{n}{3}$, and $H^1(\mathbf{X}_n,\mathbb{Z})$ has rank $\sum_{i=3}^n \binom{n}{i}\binom{i-1}{2}$.

The symmetric group $\mathbf{S}_n$ has a presentation by generators $p_1,...,p_{n-1}$ and defining relations

$$(1) \quad p_i^2 = 1, \quad 1 \leq i \leq n-1$$

$$(2) \quad p_ip_j = p_jp_i, \quad |i-j| > 1, \quad 1 \leq i,j \leq n-1$$

$$(3) \quad p_ip_{i+1}p_i = p_{i+1}p_ip_{i+1} \quad 1 \leq i \leq n-2.$$

For $j = 1,2,3$ denote by $G_{n,j}$ the group with generators $p_1,...,p_{n-1}$ and defining relations $(1)-(3)$ with relations (j) relaxed. For $j = 1,2,3$ the mapping $p_i \to (i,i+1), 1 \leq i \leq n-1$ extends to a homomorphism of $G_{n,j}$ to the symmetric group $\mathbf{S}_n$. Denote by $G^0_{n,j}$ the kernel of this homomorphism.

Proposition 3.2.

(a) $G_{n,1}$ is the braid group on n strings, $G^0_{n,1}$ is the pure braid group on n strings. The fundamental group of the $K(\pi,1)$ space $\mathbb{C}^n \setminus \{$diagonals $z_i = z_j\}$ is isomorphic to $G^0_{n,1}$.

(b) $G^0_{n,2}$ is isomorphic to the fundamental group of the $K(\pi,1)$ space $\mathbf{Y}_n$.

(c) $G_{n,3}$ is the twin group, $G^0_{n,3}$ is the pure twin group and is isomorphic to the fundamental group of the $K(\pi,1)$ space $\mathbf{X}_n$.

Part (a) is well-known, (c) is proposition 3.1, (b) is proved similar to proposition 3.1 but now instead of twins we consider configurations of n monotonic arcs without 4 arcs intersecting in a point and without two multiple points on the same horizontal line. Connected components of the space of such configurations are called *triplets.* Triplets on n strings constitute a group isomorphic to $G_{n,2}$. Pure triplet group $G^0_{n,2}$ is isomorphic to the fundamental group of $\mathbf{Y}_n$.
□

Next we describe fundamental groups of $\mathbf{Q}_n, \mathbf{Q}_{n,V_n(4)}, \mathbf{Q}_{n,V_n(3)}$.

Consider the topological space of configurations of $2n$ continuous arcs in the strip $\mathbb{R} \times [0,1]$ between the lines $y=0$ and $y=1$ such that

(i) arcs connect points $(\pm 1, 1), ..., (\pm n, 1)$ with $(\pm 1, 0), ..., (\pm n, 0)$ (in some order),
(ii) arcs descent monotonically and never go upward (i.e. each arc intersects each line $y = c$ $(0 \leq c \leq 1)$ in one point),
(iii) configuration is symmetric relative to the line $x = 0$,
(iv) no three arcs have a common point.

Denote the space of all such configurations by $\mathcal{J}_n(1)$.
Denote by $\mathcal{J}_n(2)$ the subspace of $\mathcal{J}_n(1)$ consisting of configurations with no three double points with non-zero x-coordinate on the same horizontal line.
Denote by $\mathcal{J}_n(3)$ the subspace of $\mathcal{J}_n(1)$ consisting of configurations such that a double point with zero x-coordinate is not on the same horizontal line with another double point.
Denote by $\mathcal{J}_n(4)$ the space of configurations satisfying (i)-(iii) and extra two conditions
(v) no three arcs have a common point unless it is a point on the line $x = 0$,
(vi) no 5 arcs have a common point
(note that $\mathcal{J}_n(4)$ is not a subspace of $\mathcal{J}_n(1)$).

For $j = 1, 2, 3, 4$ denote by $\Psi_n(j)$ the set of connected components of the space $\mathcal{J}_n(j)$. For $j = 1, 2, 3, 4$ the set $\Psi_n(j)$ has a group structure given, as in the case of twins, by the concatenation of diagrams.

The Weyl group of the root system B_n has a presentation by generators $\sigma, \sigma_1, ..., \sigma_{n-1}$ and relations

$$
\begin{aligned}
&(1) \quad \sigma^2 = \sigma_1^2 = ... = \sigma_{n-1}^2 \\
&(2) \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1 \\
&(3) \quad \sigma \sigma_i = \sigma_i \sigma \text{ for } i \neq n - 1 \\
&(4) \quad (\sigma \sigma_{n-1})^4 = 1 \\
&(5) \quad (\sigma_i \sigma_{i+1})^3 = 1
\end{aligned}
$$

Proposition 3.3. *Groups $\Psi_n(j), j = 1, 2, 3, 4$ are presented by generators $\sigma, \sigma_1, ..., \sigma_{n-1}$ and some of the relations $(1) - (4)$ above. Precisely,*
relations $(1), (2), (3)$ for $\Psi_n(1)$,
relations (1) and (3) for $\Psi_n(2)$,
relations (1) and (2) for $\Psi_n(3)$,
relations $(1), (2), (3), (4)$ for $\Psi_n(4)$.

The configurations defining these generators are depicted on Figure 5.

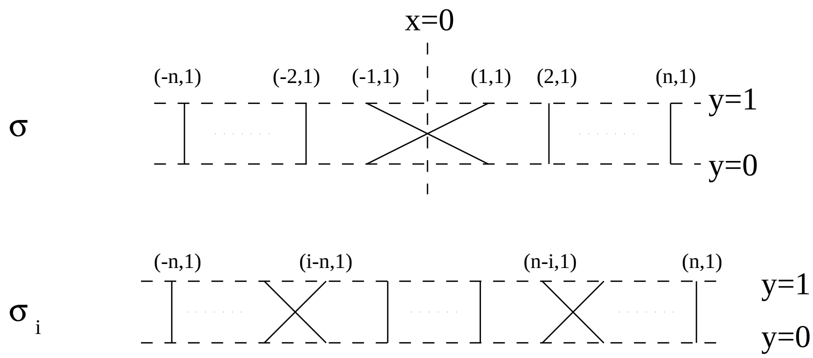

Figure 5

Denote by $\Psi_n^0(j)$ the kernel of the natural homomorphism from $\Psi_n(j)$ to the Weyl group of the root system B_n.

Denote by $\mathbf{D}_{n,3}$ the arrangement of codimension two subspaces of $\mathbb{R}^n$ given by the equations

$$x_i = \epsilon x_j = \delta x_k, \quad 1 \le i < j < k \le n, \quad \epsilon, \delta \in \{+1, -1\} \tag{3.1}$$

$\mathbf{D}_{n,3}$ is one of the arrangements studied in [BS].

The following proposition is proved is the same way as proposition 3.1.

Proposition 3.4. *The fundamental group of the space* $\mathbf{Q}_n$ *(respectively* $\mathbf{Q}_{n,V_n(4)}$, $\mathbf{Q}_{n,V_n(3)}$, $\mathbb{R}^n \setminus \mathbf{D}_{n,3}$*) is isomorphic to* $\Psi_n^0(1)$ *(respectively* $\Psi_n^0(2)$, $\Psi_n^0(3)$, $\Psi_n^0(4)$.*)*

The spaces $\mathbf{Q}_n, \mathbf{Q}_{n,V_n(4)}, \mathbf{Q}_{n,V_n(3)}$ are $K(\pi,1)$ spaces (corollary 1.5). We do not know how to answer the question of Anders Bjorner ([B2]) whether $\mathbb{R}^n \setminus \mathbf{D}_{n,3}$ is a $K(\pi,1)$ space.

The fundamental groups of spaces $\mathbf{X}_{n,V_n(4)}$ and $\mathbf{Q}_{n,M_n}$ admit similar (to propositions 3.1-3.4) descriptions but we omit them as we already know from §2 that these groups are free.

4. Finite root systems

Let Λ be a finite irreducible root system in a Euclidean space $\mathbb{R}^n$. Thus, Λ is one of $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$. Let W be the Weyl group and $\{H_1, ...H_k\}$ be the set of reflecting hyperplanes of Λ.

Let $\{L_1, ..., L_m\}$ be the set of codimension two subspaces of $\mathbb{R}^n$ that are intersections of pairs of reflecting hyperplanes of Λ.

Take a non-empty subset $\{L_{i_1}, ..., L_{i_s}\}$ of the set $\{L_1, ..., L_m\}$ such that the union $L_{i_1} \cup ... \cup L_{i_s}$ is invariant under the action of the Weyl group. That is, $w(L_{i_1} \cup ... \cup L_{i_s}) = L_{i_1} \cup ... \cup L_{i_s}$ for any $w \in W$.

Conjecture 4.1. *For any* $\Lambda, L_{i_1}, ..., L_{i_s}$ *as above,* $\mathbb{R}^n \setminus (L_{i_1} \cup ... \cup L_{i_s})$ *is a* $K(\pi, 1)$ *space.*

In the case Λ is the root system $\mathbb{A}_n$, the space $\mathbb{R}^n \setminus (L_{i_1} \cup ... \cup L_{i_s})$ is one of $\mathbf{X}_n, \mathbf{X}_{n,V_n(4)}, \mathbf{Y}_n$. Thus, for $\Lambda = \mathbb{A}_n$ the conjecture follows from the results of this paper.

When Λ is one of $\mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$, theorem 1.4 for $r = 0$ and $S \in \{V_3(n), V_4(n), M_n\}$ implies the conjecture for some of $\{L_{i_1}, ..., L_{i_s}\}$.

Case $\Lambda = \mathbb{G}_2$ is trivial, case $\Lambda = \mathbb{F}_4$ follows from the Sphere theorem [P].

In the case when $\{L_{i_1}, ..., L_{i_s}\}$ is the whole set $\{L_1, ..., L_m\}$, the conjecture follows from the discussion preceding theorem 2.1.

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