1. Consider the monoid $M$ of all maps from the set $X = \{0, 1\}$ to itself.
(a) Is $M$ commutative? In other words, is it true that $ab = ba$ for any $a, b \in M$?
(b) List all invertible elements of $M$ and identify the group $M^*$.

2. (a) State the definition of an ideal in a ring.
(b) Prove that the sum $I + J = \{i + j : i \in I, j \in J\}$ of ideals $I, J$ of $R$ is an ideal of $R$.
(c) Which of the following are ideals?
(I) Polynomials in $\mathbb{Z}[x]$ with all coefficients divisible by 3.
(II) Polynomials in $\mathbb{Z}[x]$ with zero constant and linear terms (so they have the form $a_2x^2+a_3x^3+\ldots$ with $a_2, a_3, \ldots \in \mathbb{Z}$).
(III) Polynomials $a_0 + a_1x + \ldots + a_nx^n$ in $\mathbb{Z}[x]$ with even $a_0$ and $a_1$ divisible by 3.

3. (a) Given a ring $R$ and ideals $I, J$ such that $J \subset I$, explain how to define a homomorphism $R/J \rightarrow R/I$. Describe your homomorphism using the language of residues in the specific case when $R = \mathbb{Z}$ is the ring of integers, $J = (4)$ and $I = (2)$.
(b) Under what conditions on $I, J$ is this map (A) surjective, (B) injective?

4. (a) Prove that there are no homomorphisms from the
four-element field \( \mathbb{F}_4 = \mathbb{F}_2[y]/(y^2 + y + 1) \) to \( \mathbb{Z}/4 \).

(b) Find all homomorphisms from \( \mathbb{Z}/4 \) to \( \mathbb{F}_4 \).

5. Compute the greatest common divisor of polynomials 
   \( f(x) = x^4 + x^2 + 1 \) and \( g(x) = x^3 + 1 \) over \( \mathbb{F}_2 \). Is the greatest common divisor an irreducible polynomial? Use your computation to factor \( f(x) \) and \( g(x) \) into products of irreducible polynomials.

6. (a) State the definition of a maximal ideal in a ring.

   (b) Prove that in any finite ring \( R \) there exists a maximal ideal. (Hint: Start with some ideal you know exists. What do you do if this ideal is not maximal?)

Extra credit: Give an example of a commutative ring \( R \) and ideals \( I, J \) such that \( R/(I + J) \) is a field, but \( R/I \) is not an integral domain.