Modern Algebra II, Spring 2015

Homework 7, due Wednesday March 25


1. Which of the following statements are true? (By algebraic number below we mean a complex number that is algebraic over $\mathbb{Q}$).
   (a) The sum of an algebraic and a transcendental complex number is always transcendental.
   (b) The product of two transcendental numbers is always transcendental.
   (c) An element of the field $F(x)$ of rational functions is transcendental over $F$ if and only if it is not in $F$.
   (d) For a field extension $E/F$, if $\alpha, \beta \in E$ are transcendental over $F$ then fields $F(\alpha), F(\beta)$ are isomorphic.
   (e) Any polynomial over the field $\mathbb{Q}$ is separable.

2. Prove that an $n$-th root $\sqrt[n]{\alpha}$ of an algebraic number $\alpha \in \mathbb{C}$ is algebraic. (By an $n$-th root of a complex number $z$ we mean any complex number $w$ such that $w^n = z$. If $z \neq 0$, there are $n$ such numbers. A complex number is algebraic if it’s a root of a polynomial $x^m + a_{m-1}x^{m-1} + \cdots + a_0$ with rational coefficients.)

3. Let $f(x) \in F[x]$ be an irreducible polynomial of degree $n$, and let $E/F$ be a splitting field of $f(x)$.
   (a) Prove that $n \mid [E : F]$.
   (b) Prove that if $f(x)$ is separable, $n \mid |Gal(E/F)|$.

4. Consider the splitting field $E$ of $x^4 - 3$ over $\mathbb{Q}$. Determine the degree of the extension $[E : \mathbb{Q}]$. Mimic the arguments we used in class in a similar problem for the polynomial $x^4 - 2$ (Start with a real root of this polynomial, consider the extension generated by that root, determine its degree over $\mathbb{Q}$ and see whether other roots of $x^4 - 3$ are missing from that extension). What can you say about the Galois group $Gal(E/\mathbb{Q})$?

5. Let $t$ be a formal variable and $\mathbb{F}_p(t)$ the field of rational functions in $t$ with coefficients in $\mathbb{F}_p$. Prove that the polynomial $x^p - t$ is irreducible in $\mathbb{F}_p(t)$. (Hint: use the field extension of $\mathbb{F}_p(t)$ we introduced in class in the example of inseparability.)