1. Which of the following statements are true? Give a brief justification for each answer.
(a) Given $E \supset B \supset F$ a chain of field extensions, if $E/F$ is algebraic then $B/F$ is algebraic.
(b) Field extension $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$ is algebraic.
(c) Extension $E/E$ is algebraic for any field $E$.
(d) If field $F$ is a countable set and extension $E/F$ is algebraic, then field $E$ is a countable set.
(e) Field extension $\mathbb{C}/\mathbb{R}$ is algebraic.

2. Suppose we are given field extensions $E/B$ and $B/F$ and the degree $[E:F]$ is a prime number. Show that either $B = E$ or $B = F$.

3. Suppose $p$ and $q$ are distinct prime numbers. Let $\alpha = \sqrt[p]{p} + \sqrt[q]{q}$. Generalize arguments done in class for $p = 2$ and $q = 3$ to show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a degree 4 extension that contains $\sqrt[p]{p}$ and $\sqrt[q]{q}$. Write down the irreducible polynomial for $\alpha$ over $\mathbb{Q}$. (You can use that $\sqrt[q]{q} \notin \mathbb{Q}(\sqrt[p]{p})$ and vice versa.)

4. Determine the minimum (irreducible) polynomial of $\sqrt{2} + \sqrt{5}$ over (a) $\mathbb{Q}$, (b) $\mathbb{Q}[\sqrt{2}]$, (c) $\mathbb{Q}[\sqrt{5}]$, (d) $\mathbb{Q}[\sqrt{2}, \sqrt{5}]$, (e) $\mathbb{R}$.

5. Factor polynomial $x^{10} + 2x^5 + 3$ into irreducible polynomials over the the field $\mathbb{F}_5$. (Hint: review class material on separability).

6. The element $1 + \sqrt{2} + \sqrt{3} + \sqrt{6}$ belongs to the field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Compute its multiplicative inverse. (Hint: start by computing multiplicative inverse of $1 + \sqrt{2}$ in the field $\mathbb{Q}(\sqrt{2})$ and the inverse of $1 + \sqrt{3}$ in the field $\mathbb{Q}(\sqrt{3})$. How are elements $1 + \sqrt{2}, 1 + \sqrt{3}, 1 + \sqrt{2} + \sqrt{3} + \sqrt{6}$ related?)

7. Show that $f(x) = x^3 + x + 1$ is irreducible over $\mathbb{Q}$. Let $\alpha$ be a root of $f$ in $\mathbb{C}$. Express

$$\frac{1}{\alpha} \quad \text{and} \quad \frac{1}{\alpha + 2}$$
as linear combinations of \( \{1, \alpha, \alpha^2\} \).

**Additional practice problems, won’t be graded:**

1. Prove the following technical lemma we used to show that the set of algebraic complex numbers \( \mathbb{A} \) is countable:
   If \( f : X \rightarrow Y \) is a map of sets, set \( Y \) is countable, and \( f^{-1}(y) \) is a finite set for any \( y \in Y \), then \( X \) is countable.
   If you need a refresher on sets and cardinalities, consult Hammack’s ”Book of proof”; there is a link to it on our website for Modern Algebra I.

2. Show that the polynomial \( x^3 + 2x^2 - 3x + 5 \) is irreducible over \( \mathbb{Q} \). (Hint: same methods as in problem 3 homework 5).

Howie, pages 63-64, problems 3.4-3.6, 3.8-3.11, 3.13, 3.14, 3.15,