1. (a) Define what it means for a group $G$ to act on a ring $R$. Prove that the set of $G$-invariants (the set of $G$-fixed elements) 

$$R^G = \{ a \in R : g(a) = a \ \forall g \in G \}$$

is a subring of $R$.

(b) Generator $g$ of the cyclic group $C_2$ acts on $\mathbb{Z}[x]$ by taking $x$ to $-x$. Explain why this indeed extends to an action of $C_2$ on $\mathbb{Z}[x]$. Can you write down this action succinctly, in terms of action on an arbitrary polynomial $f(x)$? Determine the invariant subring of $\mathbb{Z}[x]$ with respect to this action.

(c) Generator $g$ of the cyclic group $C_n$ acts on $\mathbb{C}[x]$ by identity on all $z \in \mathbb{C}$ and takes $x$ to $\omega x$, where $\omega$ is an $n$-th primitive root of unity. How goes $g$ act on $x^k$? Determine the $C_n$-invariant subring of $\mathbb{C}[x]$ with respect to this action.

2. Draw all partitions of 5 and determine the lexicographic order on these partitions.

3. For 2 variables $x_1, x_2$ write down monomial symmetric functions $m_{(3,1)}$ and $m_{(4,0)}$. Express these functions as polynomials in elementary symmetric functions $s_1$ and $s_2$. Write functions $s_1^3, s_1^4, s_1^5$ as linear combinations of monomial symmetric functions $m_{\lambda}$ for partitions $\lambda$ of 3, 4, 5 respectively.

4. For 3 variables $x_1, x_2, x_3$ express functions $s_1^3, s_1 s_2$ and $s_3$ as linear combinations of monomial functions $m_{\lambda}$, for $\lambda$ a partition of 3. Then express $m_{\lambda}$, for each partition $\lambda$ of 3, as a linear combination of $s_1^3, s_1 s_2, s_3$.

5. (also see Remark 4.5 in Friedman on page 7) Suppose $f(x) \in \mathbb{R}[x]$ is a cubic polynomial with real coefficients. Explain why $f$ is reducible over $\mathbb{R}$. Show that the discriminant of $f$ is strictly positive (a real number greater than 0) if and only if $f$ has three distinct real roots.

6. For each of the following cubic polynomials determine whether it is irreducible over $\mathbb{Q}$ and find its Galois group (you should determine
the Galois group even for reducible polynomials).
(a) \( x^3 - 3x - 1 \)
(b) \( x^3 - 2x + 1 \)
(c) \( x^3 - 4x - 1 \)
(d) \( x^3 - 7x - 6 \)