Modern Algebra I, fall 2014

Practice quiz solutions

Mark the squares that are followed by correct statements.

1. ■ Composition of two injective maps is injective.
   True

2. □ There exists only one set with one element.
   False. For instance, any integer \( n \) gives rise to a one-element set \( \{n\} \); they are distinct sets for different \( n \).

3. ■ The empty set is a subset of any set.
   True

4. ■ Given any surjective map \( f : A \to B \) of sets, subsets \( f^{-1}(b) \), over all \( b \in B \), constitute a partition of set \( A \).
   True

5. □ \((A \setminus B) \cup (B \setminus A) = A \cup B\) for any subsets \( A, B \) of a universal set \( X \).
   False. For a counterexample, choose any sets \( A \) and \( B \) with nonempty intersection.

6. □ 57 is a prime number.

7. ■ Any common divisor of natural numbers \( n \) and \( m \) divides the greatest common divisor \( \gcd(n, m) \).
   True. Decompose \( n \) and \( m \) as products of primes to see this.

8. ■ There are positive integers \( n \) and \( m \) such that \( \gcd(n, m) = \text{lcm}(n, m) \).
   True. Just take any \( n = m \).

9. □ Two consecutive numbers \( n, n+1 \) for \( n > 2 \) can both be primes.
False. One of them will be even and strictly greater than 2, hence not a prime.

10. ■ The set of strictly positive real numbers $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$ is a group under multiplication.

True

In problems below, $e$ denotes the identity of a group $G$, and $g, h$ denote elements of $G$.

11. ■ $g^{-1} = e$ if and only if $g = e$.

True, since the inverse of identity is the identity.

12. ■ Equality $hg = g$ implies that $h = e$.

True. Write $g = eg$, convert $hg = g$ to the equality $hg = eg$, and use the right cancellation law to cancel $g$.

13. □ $g$ is the identity if and only if $g^{-1} = g$.

False, condition $g^{-1} = g$ is equivalent to the condition $g^2 = e$ which is weaker than $g = e$. Any reflection $r$ satisfies $r^2 = e$, but it’s not the identity.

14. □ The symmetric group $S_3$ is abelian.

False. In class, we gave examples of two elements of $S_3$ that do not commute.