Modern Algebra I, Fall 2014

Homework 5, due Wednesday October 15 before class.

Read Sections 5.2 (Dihedral groups), 6.3 (Fermat’s and Euler’s theorems), 9.1 (isomorphism of groups), 9.2 (direct products), 10.1 (Factor groups and normal subgroups), 11.1 (homomorphisms).

1. (a) Explain why the dihedral group $D_3$ is isomorphic to the symmetric group $S_3$.
   
   (b) We saw in class that symmetries $r$ and $s$ (rotation by angle $\frac{2\pi}{3}$ counterclockwise and reflection in the x-axis, respectively) generate $D_n$ and satisfy the relations $r^n = 1$, $s^2 = 1$, $sr = r^{-1}s$. Use these relations to simplify the elements $r^5s^{-3}r^{-1}sr$, $s^5r^{-3}s^2rs$, $(sr)^{10}$ in $D_4$.
   
   (c) Show that the subgroup $H$ generated by $r$ (the subgroup of all rotations) is normal in $D_n$. What is the index of this subgroup?

2. Suppose that $p$ is a prime and $p \equiv 3 (\text{mod } 4)$, that is, we can write $p = 4n + 3$ for some nonnegative integer $n$. What is the order of the group $\mathbb{Z}_p^*$ of invertible residues modulo $p$? Suppose there exists a residue $x$ modulo $p$ such that $x^2 \equiv -1 (\text{mod } p)$. What would be the order of $x$ in the group $\mathbb{Z}_p^*$? Derive a contradiction and show that, with $p$ as above, equation $x^2 \equiv -1 (\text{mod } p)$ has no solutions.

3. Let $G$ be a cyclic group of order $n$. Show that there are exactly $\phi(n)$ generators for $G$. (Recall that $\phi(n)$ counts the number of integers $r$ coprime to $p$ and $0 < r < n$).

4. The group $\mathbb{C}^*$ of nonzero complex numbers under multiplication contains subgroups $T$ (complex numbers of absolute value 1) and $\mathbb{R}_{>0}$ (positive real numbers). Check that $\mathbb{C}^*$ is the internal direct product of these two subgroups.

5. (a) Prove that groups $\mathbb{Z}$ and $n\mathbb{Z}$ are isomorphic (group operation is addition for both groups).
   
   (b) Prove that $\mathbb{Q}$ is not isomorphic to $\mathbb{Z}$ (group operation is addition). Hint: can you show that $\mathbb{Q}$ is not cyclic?
   
   (c) Let $H_1$ and $H_2$ be subgroups of $G_1$ and $G_2$, respectively. Explain why $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.

6. (a) Explain why groups $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are not isomorphic.
(b) Same for groups $\mathbb{Z}_6$ and $S_3$.
(c) Same for groups $S_4$ and $A_5$.
(d) Same for groups $A_4$ and $D_6$.

7. For each of the following groups $G$, determine whether $H$ is a normal subgroup of $G$.
   (a) $G = S_4$ and $H = A_4$.
   (b) $G = A_5$ and $H = \{\text{id}, (123), (132)\}$.
   (c) $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$.

8. Give an example of a normal subgroup $H$ of $G$ such that $H$ and $G/H$ are abelian but $G$ is not.