Name:

Modern algebra I, Final exam, November 17

Textbooks, laptops, cell phones, notes are not allowed on the exam. All problems are worth the same number of points. You can write solutions to problems in any order. Good luck!

1. Mark the boxes that are followed by correct statements.

□ Composition of two injective maps is injective.
□ Group \((\mathbb{Z}/6)^*\) is cyclic.
□ Any inner automorphism of \(S_3\) is trivial.
□ The inverse of an automorphism is an automorphism.
□ Any action of the group \(C_{10}\) on a 10-element set is transitive.

2. Which of the following are equivalence relations on the set of permutations in \(S_5\)?
   (a) \(\sigma \sim \tau\) iff \(\sigma^2 = \tau\).
   (b) \(\sigma \sim \tau\) iff there exists permutation \(u\) such that \(\tau = u\sigma u^{-1}\).
   (c) \(\sigma \sim \tau\) iff \(\sigma\tau = \tau\sigma\).

3. Which of the following pairs (set, binary operation) are groups? Explain your answer.
   (a) The set \(M_n(\mathbb{R})\) of \(n\)-by-\(n\) matrices with real coefficients under addition.
   (b) The set \(\mathbb{Q}\) of rational numbers with binary operation \(x * y = x + y + 2\).
   (c) The set of 2-by-2 invertible matrices with real coefficients and positive determinant under multiplication.

4. (a) Determine cycle presentation for permutations
    \((135)(145)(125), \quad (1534)^{-78}, \quad ((143)(12))^{21}\).
(b) Which permutations in (a) are odd and which are even?
(c) Write (1234) as a product of transpositions.

5. Can you realize $C_9$ as a subgroup of $S_5$? (In other words, is $C_9$ isomorphic to a subgroup of $S_5$?) Why? What about $C_6$?

6. What does the conjugacy class of $\sigma = (123)$ in $S_4$ consist of? How many elements are in this conjugacy class? Use this information to find the centralizer of $\sigma$ in $S_4$.

7. Describe all homomorphisms (a) from $S_3$ to $C_\infty$ (the infinite cyclic group), (b) from $C_2$ to $C_6$. For each homomorphism determine its kernel and image.

8. (a) Define what a Sylow $p$-subgroup is and state the Sylow Theorems.
(b) Classify groups of order 175.

9. Let $G$ be a finite group, $H$ a Sylow 2-subgroup of $G$ and $K$ a Sylow 3-subgroup of $G$. Show that $H \cap K = 1$.

10. Consider the group $G$ of rigid motions of the regular tetrahedron. What is the order of $G$? This group acts on the set of vertices and on the set of edges of the tetrahedron. Which of these two actions are transitive? Choose a vertex and describe the stabilizer subgroup of this vertex. For the action of $G$ on vertices, what are possible numbers of fixed points of $g \in G$?

**Optional problem (extra credit):** Prove that any homomorphism from the group $C_n \times C_n$, for $n > 1$, to the circle group $\mathbb{T}$ (the group of unit complex numbers) has a nontrivial kernel.