Solutions for Midterm exam 2

1. (10 points) Mark the boxes that are followed by correct statements.

□ The intersection $H \cap K$ of normal subgroups $H$ and $K$ of $G$ is normal in $G$.

**True.** The intersection $H \cap K$ is a subgroup of $G$, and $g(H \cap K)g^{-1} = (gHg^{-1}) \cap (gKg^{-1}) = H \cap K$, implying normality.

□ Any abelian group of order 8 contains an element of order 4.

**False.** $C_2 \times C_2 \times C_2$ is abelian of order 8, but without an order 4 element.

□ The centralizer of $(12)$ in $S_3$ has order 3.

**False.** The centralizer is an order two group, $\{\text{id}, (12)\}$.

□ Any group of order 10 is abelian.

**False.** Dihedral group $D_5$ is not abelian.

□ If an abelian group $G$ has an element of order 3, then $G$ has at least 3 characters.

**True.** Number of characters equals the order of finite abelian group.

2. (10 points) Classify homomorphisms from (a) the cyclic group $C_4$ to $S_3$; (b) the cyclic group $C_5$ to $S_3$; (c) the group $S_3$ to $C_2$. Explain why these are all the homomorphisms.

Homomorphisms from $C_n$ to $H$ correspond to elements $h$ of $H$ with $h^n = 1$. For (a), there are four homomorphisms, taking a generator $g$ of $C_n$ to id, $(12)$, $(13)$, $(23)$, correspondingly. For (b), there is only the trivial homomorphism, since the only permutation $\sigma$ in $S_3$ with $\sigma^5 = 1$ is the trivial permutation id. For (c), any homomorphism from $S_3$ to an abelian group will take the commutator subgroup $A_3 = S_3'$ to identity. This gives only two homomorphisms from $S_3$ to $C_2$: either take all $S_3$ to the unit element, or take even permutations to the unit element and all odd ones to the nontrivial element of $C_2$.

3. (10 points) For each of the following groups $G$ determine whether $H$ is a normal subgroup of $G$.

(a) $G = S_4$ and $H \cong S_3$ is the subgroup of permutations that fix 4.

Not normal. We need to find an element $h$ of $H$ and element $g$ of $G$ such that $ghg^{-1}$ is not in $H$. For instance, take $h = (12)$ and $g = (14)$. Then
\[ g h g^{-1} = (14)(12)(14)^{-1} = (14)(12)(14) = (24), \text{ not in } H. \]

(b) \( G = A_4 \) and \( H = \{1, (12)(34)\} \).

Not normal. Conjugate \((12)(34)\) by something in \( A_4 \), for instance \((123)\), to get
\((123)(12)(34)(132) = (14)(23)\), which is not in \( H \).

(c) \( G = D_n \), the dihedral group, and \( H = C_n \), the subgroup of rotations in \( G \).

Normal, for instance since \( C_n \) has index 2 in \( D_n \), and any index 2 subgroup is normal.

4. (10 points) Write down the character table of the group \( C_2 \times C_2 \).

Characters of finite abelian groups are homomorphisms to the circle group \( T \). The number of homomorphisms equals the order of the group, in this case 4. Group \( C_2 \times C_2 \) consists of elements of order 1 and 2 only, so under a homomorphism these elements can only go to the elements 1 and \(-1\) of \( T \). If we denote generators of \( C_2 \times C_2 \) by \( a \) and \( b \), we have four choices: send \( a \) either to 1 or \(-1\), and, independently, send \( b \) either to 1 or \(-1\). Then \( ab \) will go to the corresponding product, and 1 always goes to one. The character table is

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>1</th>
<th>( a )</th>
<th>( b )</th>
<th>( ab )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_0 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>(-1)</td>
<td>1</td>
<td>(-1)</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>1</td>
<td>1</td>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>1</td>
<td>(-1)</td>
<td>(-1)</td>
<td>1</td>
</tr>
</tbody>
</table>

Columns are labelled by elements of the groups, rows - by characters. The first character \( \chi_0 \) is the trivial character.

5. (10 points) Write down a proof that, given two subgroups \( K, H \) of \( G \) with \( K \) normal, the set \( KH = \{kh : k \in K, h \in H\} \) is a subgroup of \( G \).

The set \( KH \) contains the unit element, since \( 1 = 1 \cdot 1 \in K \cdot H \). It’s closed under taking inverses: if \( k \in K, h \in H \), then

\[(kh)^{-1} = h^{-1}k^{-1} = (h^{-1}k^{-1}h)h^{-1}.\]

Since \( h^{-1}k^{-1}h \) is in \( h^{-1}Kh = K \), by normality of \( K \), we see that \((kh)^{-1} \in KH \). Finally, \( KH \) is closed under taking products: given \( k_1, k_2 \in K, h_1, h_2 \in K \), \( (k_1h_1)(k_2h_2) = k_1(k_2h_1h_2) \in KH \).
so that \( k_1 h_1, k_2 h_2 \in KH \), we have

\[
k_1 h_1 \cdot k_2 h_2 = k_1 (h_1 k_2 h_1^{-1})(h_1 h_2) \in KH,
\]

since \( k_1 (h_1 k_2 h_1^{-1}) \in KK = K \) and \( h_1 h_2 \in H \).

6. (20 points) (a) Find all subgroups of \( Q_8 \).

\( Q_8 \) has order 8, and can only have subgroups of orders 1, 2, 4, 8. The trivial subgroup has order 1, \( Q_8 \) itself has order 8. Elements \(-1, i, j, k\) each generate a subgroup, of order 2, 4, 4, 4, respectively:

- \( \langle -1 \rangle = \{1, -1\} \).
- \( \langle i \rangle = \{1, i, -1, -i\} \).
- \( \langle j \rangle = \{1, j, -1, -j\} \).
- \( \langle k \rangle = \{1, k, -1, -k\} \).

It’s easy to see that there are no other subgroups. For instance, if a subgroup \( H \) contains at least one of the six elements \( \{\pm i, \pm j, \pm k\} \) of order four, it has order at least four, so it’s either the cyclic group generated by that element or all of \( Q_8 \). If it contains none of these six elements, it’s either trivial of the center \( \{1, -1\} \) of \( Q_8 \). Thus, \( Q_8 \) has six subgroups.

(b) Show that all subgroups of \( Q_8 \) are normal.

The three subgroup of order 4 are normal, since they have index 2. The center of any group is a normal subgroup in it. The trivial group and all of \( Q_8 \) are normal.

(c) Write down definition of a solvable group.

\( G \) is solvable if there is a chain of subgroups \( G = G_n \supset G_{n-1} \supset \cdots \supset G_1 \supset G_0 = 1 \) such that \( G_{i-1} \) is normal in \( G_i \) for \( i = 1, \ldots, n \) and \( G_i/G_{i-1} \) is abelian.

(d) Use this definition to prove directly that \( Q_8 \) is solvable (We proved in class that any group of order \( p^n \) is solvable; don’t use this result).

As a chain of subgroups, we can take, for instance,

\[
Q_8 \supset \{1, i, -1, -i\} \supset \{1, -1\} \supset \{1\}.
\]

Each subgroup has index two in the previous group, thus normal in it and the quotient group has order 2, hence abelian.