Name: ________________________ UNI: ________________

Modern algebra I, Spring 2017. Practice Final Exam.

1. Which of the following are equivalence relations on the set $\mathbb{R}$ of real numbers?
(a) $x \equiv y$ iff $x - y \in \mathbb{Q}$,
(b) $x \equiv y$ iff $x \leq y$,
(c) $x \equiv y$ iff $|x| = |y|$.

1.5. Consider the group of invertible residues $\mathbb{Z}_{20}^*$. What is the order of this group? Is this group cyclic?

2. (a) State the definition of a group.
(b) Write down the multiplication table for the Klein four group V_4.

3. (a) Is the set of all even permutations in S_6 that fix 4 a subgroup of S_6? What about all odd permutations that fix 1?
(b) Is the set of all rational numbers with the binary operation $x * y = 2(x + y)$ a group?

4. (a) Simplify the following permutations and write them as products of disjoint cycles:
(a) $(1354)(264)(254)$, (b) $(1532)^{22}$, (c) $((132)(23))^{-17}$.
(b) Decompose (16354) as a product of transpositions.
(c) What are the possible cycle types of permutations in A_5? In A_6?

5. State the 2nd Isomorphism Theorem. What does it tell you for in case when the group is D_4, the normal subgroup is C_4 of all rotations in D_4, and the other subgroup is generated by the reflection s about a diagonal. For this case compute all groups that appear in the diagram describing the theorem.

6. Describe all homomorphisms (a) from S_3 to C_4, (b) from C_4 to $C_2 \times C_2$, (c) from $C_2 \times C_2$ to S_3.

7. If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \times G/K$.

8. Prove that any subgroup of Q_8 is normal. Are there any groups of order 8 with a non-normal subgroup?

9. List all isomorphism classes of abelian groups of order (a) 16, (b) 40.

10. Can you realize C_7 as a subgroup of S_7? of S_6? of A_7?

11. (a) Define the notion of a Sylow p-subgroup. State the Sylow Theorems.
(b) Classify groups of order 175.
(c) Give an example of a Sylow 2-subgroup of A_5.

12. How any elements are in the conjugacy class of (234) in S_5? What is the centralizer of that element in S_5?

13. In how many ways can you color the faces of a cube using 5 colors, up to rotational symmetries? Use the counting theorem.

14. What is the order of the group $Rot(C)$ of rotations of the cube? This group acts on the set of faces of the cube. Is this a transitive action? Choose a face of a cube and find the stabilizer subgroup of the face. For the action of $Rot(C)$ on faces, what are possible numbers of fixed points of $g \in Rot(C)$?

15. Mark the boxes that are followed by correct statements.

☐ The empty set is a subgroup of any group G.

☐ The composition of injective group homomorphisms is an injective group homomorphism.

☐ Any homomorphism from $\mathbb{Z}$ to a finite group has a non-trivial kernel.

☐ Group $S_4 \times S_3$ has an abelian Sylow 3-subgroup.

☐ The quaternion group Q_8 is not isomorphic to a subgroup of S_5.

☐ Any solvable group is finite.