Modern Algebra I, Spring 2017

Homework 7, due Wednesday March 8 before class.
Read Section 9 (both 9.1 and 9.2), Section 10.1, and Section 11.1. Solve the following problems.

1. A homomorphism $f : G \longrightarrow H$ of groups is a map such that $f(ab) = f(a)f(b)$ for any $a, b \in G$. Prove that f takes the unit element of G to the unit element of H and that $f(a^{-1}) = f(a)^{-1}$.

2. Collect the following groups into isomorphism classes:
(a) C_4, (b) $\mathbb{Z}_6$, (c) $\mathbb{Z}_4$, (d) $\mathbb{Z}_3 \times \mathbb{Z}_2$, (e) $\mathbb{Z}_2 \times \mathbb{Z}_2$, (f) $\mathbb{Z}_5^*$, (g) $\mathbb{Z}_8^*$.
(Hint: for starters, figure out the order of each group and whether the group is cyclic. Notice that Judson denotes $\mathbb{Z}_n^*$ by $U(n)$.)

3. Show that the groups $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are pairwise non-isomorphic. (Hint: what are possible orders of elements of these groups?)

4. Prove that $\mathbb{Q}$ is not isomorphic to $\mathbb{Z}$. (Hint: assume they are isomorphic. Look at the image of a generator of $\mathbb{Z}$ under this isomorphism.)

5. Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5, respectively such that $hk = kh$ for all $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.

6. If a group G has exactly one subgroup H of order k, prove that H is normal in G.

7. (a) Let $H = \{id, (12)(34), (13)(24), (14)(23)\}$. Check that H is a subgroup of S_4. Prove that H is normal in S_4 using that two permutations are conjugate in S_n iff they have the same cycle type (proved in class).
(b) Show that the subgroup H generated by the 4-cycle (1234) is not normal in S_4.

8. Show that the intersection of two normal subgroups is a normal subgroup.

9. Judson exercise 2abd in Section 11.3 (page 130). Remember that $\mathbb{R}$ has group operation addition, and $\mathbb{R}^*$ multiplication. $GL_2(\mathbb{R})$ is the group of invertible 2-by-2 matrices under multiplication.