Modern Algebra I, Spring 2017

Homework 4, due Wednesday February 15 before class.
Read Judson Sections 4.1, 4.2 (some parts of 4.2 are just a review of complex numbers, you can skim over). Solve the following problems.

1. (a) Consider the cyclic group C_{10} with generator g. Find the orders of g, g^2, g^5, g^4, g^3. Is C_{10} commutative? Give an example of a generator of C_{10} other than g.

(b) Find and list all subgroups of C_{10}. Use the classification of subgroups of cyclic groups proved in class.

2. Consider the cyclic group C_{40} with generator a. Find the orders of elements a^2, a^{12}, a^{-5}, a^{11}. Does C_{40} have a subgroup of order 8? Of order 12?

3. Which of the following subsets of $\mathbb{C}^*$ are subgroups? (Recall that $\mathbb{C}^*$ is the group of nonzero complex numbers under multiplication.) (a) $\{1, -1\}$, (b) $\{i, -i\}$, (c) $\{z \in \mathbb{C}^* : |z| = 1\}$ (complex numbers of unit length), (d) $\mathbb{R}^*$ (real nonzero complex numbers), (e) $\mathbb{R}^* \cup i\mathbb{R}^*$ (the union of x-axis and y-axis in the complex plane, without zero).

4. (a) Consider the group $\mathbb{Z}_9^*$ of invertible residues modulo 9 under multiplication. List all elements of this group, how many are there? Pick one or more elements of this group and find their orders. Is the group cyclic (can you find a generator)?

(b) Same as (a) for the group $\mathbb{Z}_{12}^*$.

(c) Same as (a) for the group $\mathbb{Z}_{11}^*$.

5. In class we defined the direct product $G \times H$ of groups G and H to consist of elements (g, h), for $g \in G$ and $h \in H$, with the multiplication operation

$$(g_1, h_1) \circ (g_2, h_2) \ = \ (g_1 g_2, h_1 h_2).$$

(a) Prove that $G \times H$ is a group.

(b) Prove that $G \times H$ is abelian if both G and H are abelian.