Modern Algebra I, Spring 2017

Homework 11, due Wednesday April 12 before class.

Read Judson 14.1, 14.2 (Group actions, class equation) and Gallagher Section 16.

1. Solve Exercise 1 in Gallagher Section 16.

2. Dihedral group D_4 is the group of symmetries of the square. Describe orbits and stabilizer subgroups for various points in the square, similar to how we did it in class for S_3 acting on an equilateral triangle. Can you realize the rotation subgroup C_4 of D_4 as the stabilizer of a point of a square?

3. Find the conjugacy classes and the class equation for each of the following groups
(a) S_4, (b) D_5, (c) $\mathbb{Z}_6$, (d) Q_8.

4. (a) The group of integers $\mathbb{Z}$ acts on the set of real numbers $\mathbb{R}$ by adding an integer to a real number, so $n \in \mathbb{Z}$ acts on $x \in \mathbb{R}$ by taking it to $x + n$. What are the orbits of this action? What is the stabilizer subgroup $\mathbb{Z}_x$, for $x \in \mathbb{R}$?
(b) The circle group $\mathbb{T}$ acts on the plane $\mathbb{R}^2$ by rotating vectors about the center of the plane. You can also think about this as an action on complex numbers $\mathbb{C}$, with a unit complex number $e^{i\psi} \in \mathbb{T}$ acting on $z \in \mathbb{C}$ by taking it to $e^{i\psi}z$. Describe orbits of this action (it helps to draw a picture). What is the stabilizer of $0 \in \mathbb{C}$? What is the stabilizer of $1 + i \in \mathbb{C}$?

5. Assume that a cyclic group C_p of order a prime number p acts on a set X and pick an orbit of C_p in X. Show that the orbit either consists of p different points or just one point.

6. The symmetric group S_n naturally acts on the set $J = \{1, 2, \ldots, n\}$ of numbers from 1 to n.
(a) Check that $\sigma(i, j) = (\sigma(i), \sigma(j))$ defines an action of S_n on the set $J \times J$ of all pairs of elements from J.
(b) Show that S_n has exactly two orbits in $J \times J$.
(c) For each of these orbits, pick a representative (an element in the orbit) and find the stabilizer subgroup for this element.

FLIP OVER

7. Let G act on a set X. Given $A \subset X$ we can define $gA = \{ga : a \in A\}$. Thus, gA is a subset of X. Check that this is an action of G on the powerset $P(X)$ of X (the set of all subsets of X). What is the stabilizer of the empty set $\emptyset$ under this action? What is the stabilizer of a one-element set $\{x\}$, for $x \in X$?