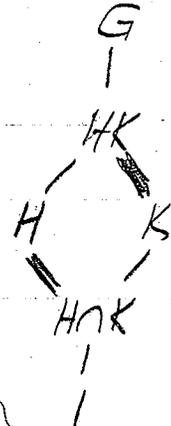


At this point please review the definitions of normal subgroup, factor group, homomorphism, image, kernel, isomorphism, isomorphic, and the statement of 1st Isomorphism Thm.

THM A (2nd Isomorphism Theorem) If  $H$  &  $K$  are subgroups of a group  $G$ , with  $K \triangleleft G$ , then

- (1)  $H \cap K$  is a normal subgroup of  $H$ ;
- (2)  $HK/K$  is a subgroup of  $G/K$ ;
- (3)  $HK/K \cong H/(H \cap K)$



Proof (1) Clearly  $1 \in HK$ . For  $h_1, h_2 \in H$  &  $k_1, k_2 \in K$ ,

$$(h_1 k_1)(h_2 k_2)^{-1} = h_1 k_1 k_2^{-1} h_2^{-1} = \underbrace{h_1 h_2^{-1}}_{\in H} \underbrace{(k_1 k_2^{-1})}_{\in K} h_2^{-1} \in HK,$$

the step  $\in$  using the fact that  $K \triangleleft G$ . Therefore  $HK$  is a subgroup of  $G$ .

(2 & 3). The canonical homomorphism sending  $g \in G$  to  $gK \in G/K$  has kernel  $K$  and image  $G/K$  (by Thm E on 11.4). Restricting this map to  $H$  gives a homomorphism  $\varphi: H \rightarrow G/K$  with kernel  $H \cap K$  and image  $HK/K$ , since

$$h \in \ker \varphi \iff h \in H \text{ and } hK = K \iff h \in H \text{ and } h \in K \iff h \in H \cap K$$

and

$$\text{im. } \varphi = \{hK : h \in H\} = HK/K.$$

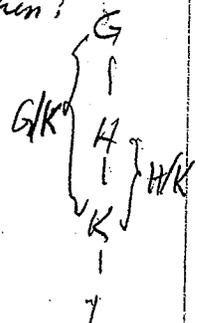
Therefore, (1), (2), (3) follow from (1), (2), (3) in the 1st Isomorphism Theorem.

THM B (Correspondence Theorem) Let  $K$  be a normal subgroup of  $G$ . Then:

(1) For each subgroup  $H$  of  $G$  containing  $K$ ,  $H/K$  is a subgroup of  $G/K$ ;

(2) The map  $H \mapsto H/K$  is a bijection

from  $\{ \text{all subgroups } H \text{ of } G \text{ containing } K \}$  to  $\{ \text{all subgroups of } G/K \}$ .



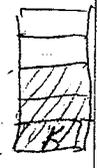
Proof (1) If  $H$  is a subgroup of  $G$  containing  $K$ , then  $HK = H$ , so  $HK/K = H/K$ , so THM A(2) implies  $H/K$  is a subgroup of  $G/K$ .

(2) By (1) the map sending  $H$  to  $H/K$  from  $\{ \dots \}$  to  $\{ \dots \}$  is well-defined.

This map is injective since if  $H_1, H_2$  are subgroups of  $G$  containing  $K$ , and  $H_1/K = H_2/K$ , then  $H_1$  and  $H_2$  consist of the same  $K$ -cosets, so  $H_1 = H_2$ .

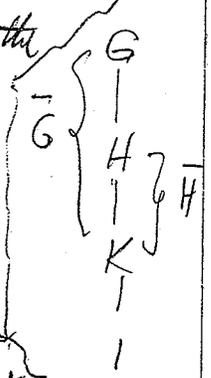
This map is surjective: In fact, let  $L$  be any subgroup of  $G/K$ .

Let  $H$  be the union of the  $K$ -cosets making up  $L$ . Then  $H$  is a subgroup of  $G$  since  $1 \in 1 \cdot K \in L$  so  $1 \in H$ ; and  $a, b \in H$  implies  $aK, bK \in L$  so  $ab^{-1}K = aK(bK)^{-1} \in L$ , so  $ab^{-1} \in H$ .



By our definition of  $H$ ,  $L$  is the set of  $K$ -cosets in  $H$ , so our map takes  $H$  to  $L$ .

LIGHTER NOTATION. A commonly used simpler notation for the canonical homomorphism  $g \mapsto gK$  from  $G$  to  $G/K$  (for  $K \triangleleft G$ ) is  $\bar{g}$  for  $gK$ ,  $\bar{G}$  for  $G/K$ , and more generally  $\bar{H}$  for  $H/K$  for each subgroup  $H$  of  $G$  containing  $K$ . Observe that  $\overline{ab} = \bar{a}\bar{b}$  for  $a, b \in G$ .



THM C (3rd Isomorphism Theorem) Let  $K \triangleleft G$  and let  $H$  be a subgroup of  $G$  containing  $K$ .

(1)  $H \triangleleft G \iff \bar{H} \triangleleft \bar{G}$ ;

(2)\*  $G/H \cong \bar{G}/\bar{H}$ , for  $H \triangleleft G$ .

Proof (1). First, assume  $H \triangleleft G$ . Then for  $h \in H$  and  $g \in G$

$$(gK)(hK)(gK)^{-1} = (ghg^{-1})K \in H/K, \text{ since } ghg^{-1} \in H.$$

Therefore  $H/K \triangleleft G/K$ .

Next, assume only  $H/K \triangleleft G/K$ . Then for  $h \in H$  and  $g \in G$ ,

$$ghg^{-1}K = (gK)(hK)(gK)^{-1} \in H/K.$$

Therefore  $H \triangleleft G$ .

\*) In the heavier notation, (2) reads  $G/H \cong (G/K)/(H/K)$ , for  $H \triangleleft G$ .

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(2) Now assume (besides  $K \triangleleft G$ ) that  $H \triangleleft G$ . Define  $\varphi: G/K \rightarrow G/H$  by

$$\varphi(gK) = gH.$$

Is  $\varphi$  really well-defined? If  $g_1K = g_2K$  then  $g_1^{-1}g_2 \in K \subset H$  so  $g_1H = g_2H$ , so, yes,  $\varphi$  is well defined.

Is  $\varphi$  a homomorphism? If  $aK$  &  $bK \in G/K$ , then

$$\varphi(aK \cdot bK) = \varphi(abK) = abH = aH \cdot bH = \varphi(aK) \varphi(bK),$$

so  $\varphi$  is a homomorphism.

For  $gH \in G/H$ , we have  $gH = \varphi(gK)$ , so  $\varphi$  is surjective, i.e.

$$\checkmark \quad \text{im } \varphi = G/H.$$

For  $gK \in G/K$ , we have  $\varphi(gK) = H \Leftrightarrow gH = H \Leftrightarrow g \in H \Leftrightarrow gH \in H/K$ , so

$$\surd \quad \ker \varphi = H/K.$$

The 1st Isomorphism Theorem, together with  $\checkmark$  and  $\surd$  now gives

$$G/H = \text{im } \varphi \cong (G/K) / \ker \varphi = (G/K) / (H/K),$$

i.e.

$$G/H \cong \overline{G/\overline{H}}.$$

REMARK Each subgroup and each factor group of an abelian group is abelian. Next we come to a more general class of groups with this property.

DEF A group  $G$  is solvable if there is a finite chain (i.e. decreasing sequence) of subgroups

$$G = G_0 \supset G_1 \supset \dots \supset G_s = 1$$

so that

$$G_k \triangleleft G_{k-1} \text{ \& } G_{k-1}/G_k \text{ is abelian, for } k=1, \dots, s.$$

EXAMPLES Each abelian group is solvable. (with  $s=1$ ).  $S_3$  is solvable (why?).

LEMMA 1. Let  $A, B, K$  be subgroups of a group  $G$ , with  $A \triangleleft B$  and  $K \triangleleft G$ .

Put  $\bar{A} = AK/K$  &  $\bar{B} = BK/K$ . Then  $\bar{A} \triangleleft \bar{B}$ , and  $\bar{B}/\bar{A}$  is isomorphic to a factor group of  $B/A$ .

Proof. One checks easily that  $AK \triangleleft BK$ . Therefore  $\bar{A} \triangleleft \bar{B}$  (by 3rd Isomorphism Theorem) and

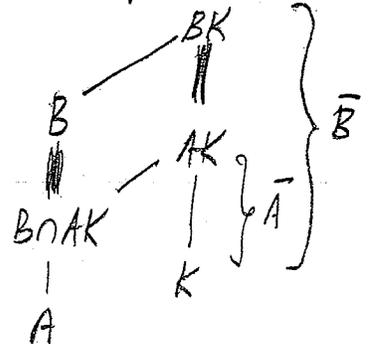
$$\bar{B}/\bar{A} = (BK)/K / (AK)/K$$

$$\cong BK/AK \quad (\text{3rd Isomorphism Thm})$$

$$= B(AK)/AK \quad (A \subset B)$$

$$\cong B/(B \cap AK) \quad (\text{2nd Isomorphism Thm \& } AK \triangleleft BK)$$

$$\cong (B/A)/(B \cap AK)/A, \quad (\text{3rd Isomorphism Thm})$$



which is a factor group of  $B/A$ .

EXERCISE 1. Prove that  $A \triangleleft B$  and  $K \triangleleft G$  imply  $AK \triangleleft BK$ .

THM D. Each factor group of a solvable group is solvable. More generally, if

$$\checkmark \quad G = G_0 \supset G_1 \supset \dots \supset G_s$$

are subgroups of  $G$  so that

$$G_k \triangleleft G_{k-1} \text{ and } G_{k-1}/G_k \text{ is abelian for } k=1, \dots, s$$

and if

$$K \triangleleft G \text{ with } G_s \subset K,$$

then  $G/K$  is solvable. (The case  $G_s = 1$  gives the first statement).

Proof. Put  $F_k = G_k K / K$  for  $k=0, 1, \dots, s$ . From  $\checkmark$  we get

$$G = GK = G_0 K \supset G_1 K \supset \dots \supset G_s K,$$

$$G/K = G_0 K / K = F_0 \supset F_1 \supset \dots \supset F_s.$$

Since  $G_5 \triangleleft K$ , we have  $F_5 = G_5 K / K = 1$  in  $G/K$ . By Lemma 1, for  $k=1, \dots, 5$

$F_k \triangleleft F_{k-1}$  and  $F_{k-1}/F_k$  is isomorphic to a factor group of  $G_{k-1}/G_k$  & is abelian.

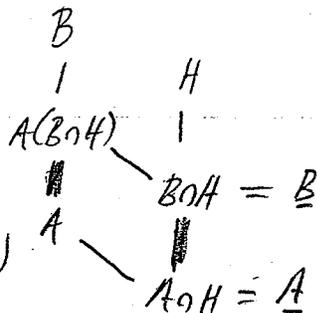
It follows that  $G/K$  is solvable.

LEMMA 2. Let  $A, B, H$  be subgroups of a group  $G$ , with  $A \triangleleft B$ .

Put  $\underline{A} = A \cap H$  &  $\underline{B} = B \cap H$ . Then  $\underline{A} \triangleleft \underline{B}$  and  $\underline{B}/\underline{A}$  is isomorphic to a subgroup of  $B/A$ .

Proof. One checks easily that  $\underline{A} \triangleleft \underline{B}$ . We have

$$\begin{aligned} \underline{B}/\underline{A} &= (B \cap H) / (A \cap H) \\ &= (B \cap H) / (A \cap (B \cap H)) \quad (A \triangleleft B) \\ &= A(B \cap H) / A, \quad (\text{2nd Isomorphism Thm}) \end{aligned}$$



which is a subgroup of  $B/A$ , by the Correspondence Theorem.

EXERCISE 2. Prove that  $A \triangleleft B$  implies  $\underline{A} \triangleleft \underline{B}$  for each subgroup  $H$  of  $G$ .

THM E. Let  $G$  be solvable. Then there are subgroups

$$G = G_0 > G_1 > \dots > G_s = 1$$

so that

$$G_k \triangleleft G_{k-1} \text{ and } G_{k-1}/G_k \text{ is abelian, for } k=1, \dots, s$$

For each subgroup  $H$  of  $G$ , put  $H_k = H \cap G_k$ , for  $k=0, 1, \dots, s$ . Then

$$H = H_0 > H_1 > \dots > H_s = 1$$

By Lemma 2, for  $k=1, \dots, s$ ,

$$H_k/H_{k-1} \text{ is isomorphic to a subgroup of } G_k/G_{k-1} \text{ & is abelian.}$$

It follows that  $H$  is solvable.

EXERCISE 3. Let  $G$  be an abelian group. Prove that for each  $n \in \mathbb{N}$

$$(1) (ab)^n = a^n b^n \text{ for all } a, b \in G.$$

$$(2) G^n \stackrel{\text{def}}{=} \{g^n : g \in G\} \text{ and } G_n = \{g \in G : g^n = 1\} \text{ are subgroups of } G.$$

$$(3) G^n \cong G/G_n.$$

EXERCISE 4. Let  $G$  be a finite abelian group of order  $n$ , and let  $G = \{g_1, \dots, g_n\}$  with any ordering of the elements of  $G$ . Put

$$w = g_1 \cdots g_n \quad (\text{the product of all the elements of } G).$$

Prove that  $w^2 = 1$ .

DEF Let  $G$  be a group and let  $a, b \in G$ . The commutator  $[a, b]$  of  $a$  &  $b$  is defined by  $[a, b] = ab a^{-1} b^{-1}$ . The commutator subgroup of  $G$ , denoted  $G'$  (or sometimes  $G^{(1)}$ ) is the subgroup generated by all commutators in  $G$ .

$$G' = \langle \{[a, b] \mid a, b \in G\} \rangle.$$

EXERCISE 5. Prove

$$(1) [a, b] = 1 \iff ab = ba$$

$$(2) G' = 1 \iff G \text{ is abelian}$$

$$(3) G' \triangleleft G \text{ and } G/G' \text{ is abelian}$$

$$(4) \text{ If } K \triangleleft G \text{ and } G/K \text{ is abelian, then } G' \subset K.$$

$$(5) G' \text{ is the intersection of all } K \triangleleft G \text{ for which } G/K \text{ is abelian.}$$

$$(6) \text{ Find } (S_3)', \text{ and } (S_3)'' (= (S_3)')'.$$

EXERCISE 6 Let  $\mathbb{R}^+$  be the additive group of all real numbers, and let  $\mathbb{P}^*$  be the subgroup of all positive numbers in  $\mathbb{R}^+$ , the multiplicative group of nonzero real numbers. Prove  $\mathbb{R}^+ \cong \mathbb{P}^*$ .