Take-home final, will be made available Sunday Dec 20 (morning)
due Mon Dec 21 4pm

Review Session: Sat, Dec. 19
10^{am}

email for OH.

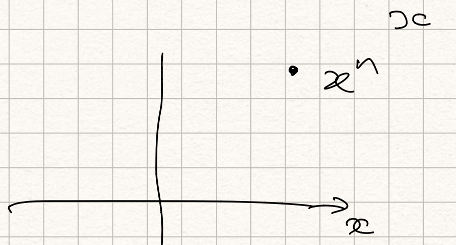
Thm $\mathbb{C}$ is algebraically closed.

Prop If $f(x) \in \mathbb{R}[x]$, $\deg(f)$ - odd, then f has a real root.

Proof

$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$

n - odd



$x >> 0$

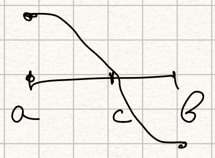
$f(x)$ $\boxed{x^n}$ $f(x) > 0$

$x << 0$

$f(x) < 0$

IVT: If $f(x)$ is continuous $[a,b]$

$f(a) f(b) < 0$



$\Rightarrow f(c) = 0$ some $a < c < b$.

polynomials are continuous

$$A = 1 + \sum_{i=0}^{n-1} |a_i|$$

$(\mathbb{R}: \ t)$

$|a_i| \leq A - 1$

$f(x)$ $\qquad f(A) = A^n + \underbrace{a_{n-1}A^{n-1} + \dots + a_0}$

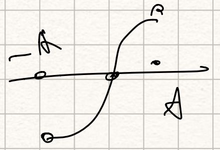
$|a_{n-1}A^{n-1} + \dots + a_0| \leq |a_{n-1}A^{n-1}| + \dots + |a_0| \leq$

$\leq |a_{n-1}| A^{n-1} + |a_{n-2}| A^{n-2} + \dots + |a_0| \leq (A-1)(A^{n-1} + A^{n-2} + \dots + 1)$

$= A^n - 1 < \underline{A^n} \implies f(A) > 0.$

$f(-A) < 0$ $\qquad (-A)^n \leftarrow$ odd

$\implies f$ has a root.



$\underline{f(x)} \in \mathbb{C}[x] \to$ want complex root.

1) $\underline{\text{reduce to } f(x)}$ $\overset{w\mathbb{R}}{\vee}$ real coefficients.

$\underline{f = \sum a_i x^i}$ $\qquad \underline{\overline{f} = \sum \overline{a_i} x^i}$ $\qquad$ (Rofman, p. 89)

$\underbrace{f\overline{f}}$ - exercise: all coefficients of $f\overline{f}$ are real

$\overline{(f(x)\overline{f}(x))} = \overline{f}(x)\,\overline{\overline{f}}(x) = \overline{f(x)}f(x).$ $\qquad \overline{x} = x$

$\overline{} : \mathbb{C} \to \mathbb{C}$ $\qquad \underline{\mathbb{C}[x] \xrightarrow{-} \mathbb{C}[x]}$ $\qquad \overline{x} = x.$

$\underline{f(x) \in \mathbb{R}[x] \text{ is irreducible}/\mathbb{R}}$ $\qquad$ want $\underline{\text{complex root.}}$

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r] & \mathbb{C} \\ x \arrow[r, mapsto] & \alpha \end{tikzcd}$$

$f(x)\overline{f}(x). = \sum_{k \geq 0} c_k x^k$ $\qquad \boxed{c_k = \sum_{i+j=k} a_i \overline{a_j}} \leftarrow$ conj. invariant

$f(x)\underbrace{(x^2+1)}$ $\qquad$ splitting field $E/\mathbb{R}$

$E \supset \mathbb{C} \supset \mathbb{R}$ $\qquad$ (E contains a subfield isomorphic to field $\mathbb{C}$)

$\uparrow$ $\qquad$ normal,

spl. field $\qquad$ separable (char 0).

$\implies E/\mathbb{R}$ $\underline{\text{Galois extension}}$ $\qquad \underline{|G| = [E:\mathbb{R}]}$

$G = \mathrm{Gal}(E/\mathbb{R})$

$|G| = 2^m \cdot \underline{k}$, k odd. $\Rightarrow \exists \, \underline{\underline{H}} \subset G$, $|H| = 2^m$

2-Sylow $\underline{\text{subgroup}}$

$E \supset E^H$ fixed field

$$\begin{array}{ccc} E & \supset & E^H \\ \cup & & \| \\ \mathbb{R} & & B \end{array}$$

$[E:B] = |H|$

$$E \supset \overset{\text{odd}}{B \supset \mathbb{R}}$$

$$\begin{array}{cc} |H| & [G:H]. \\ \| & \| \\ 2^m & k \text{ odd} \end{array}$$

$B/\mathbb{R}$ $\quad [B:\mathbb{R}] = k$

$\alpha \in B$, $\alpha \notin \mathbb{R}$ $\quad \alpha \in B \setminus \mathbb{R}$.

$$\deg \alpha = \overline{[\mathbb{R}(\alpha):\mathbb{R}]}$$

$$\| \text{ odd}$$

$$\ell \quad \text{odd}$$

$$\underbrace{B \underset{\text{odd}}{\supset} \mathbb{R}(\alpha) \underset{\text{odd}}{\supset} \mathbb{R}}_{k \text{ odd}}$$

$k = \ell \cdot [B:\mathbb{R}(\alpha)]$

$$\boxed{g(x) - \text{irred polyn } \alpha/\mathbb{R}}$$

$\deg g = \ell$ - odd

odd odd deg real. polyn. has a root. $\Rightarrow$

$\forall$ odd deg irr. real pol. is $\underline{\underline{ax+b}}$ $\quad \alpha \in \mathbb{R}$

$$\boxed{B = \underline{\mathbb{R}} \quad k=1}$$

$\underline{|G| = 2^m}$

$$\begin{array}{c} E \\ \cup 2^{m-1} \\ \mathbb{C} \\ \cup 2 \\ \mathbb{R} \end{array} \qquad G \curvearrowright 2^m$$

$[E:\mathbb{R}] = 2^m$

$\underline{L} = \mathrm{Gal}(E/\mathbb{C})$ $\qquad L \overset{2}{\subset} G$

$|L| = 2^{m-1}$ $\qquad$ find U

$[L:U] = 2$

$$\begin{array}{c} E \\ \cup \\ K \\ \cup 2 \\ \mathbb{C} \end{array} \qquad \begin{array}{c} E \\ \cup 2^{m-1} \\ \mathbb{C} \end{array} \qquad [E^U:\mathbb{C}] = 2 \qquad \begin{array}{c} E \\ \cup \\ K = E^U \\ \cup 2 \\ \mathbb{C} \end{array} \qquad \begin{array}{c} L \\ \cup 2 \\ \underline{U} \end{array}$$

$\mathbb{C} \leftarrow$ $\underline{\text{impossible,}}$ since any deg. 2 polyn/$\mathbb{C}$ is reducible

L find a subgroup U of index 2 and take its fixed field.

Claim Any group L of order p^r (prime p) has a subgroup of index p.

L is solvable, since has order p^r

$$\boxed{L \overset{p}{\supset} L^1 \overset{p}{\supset} L^2 \overset{p}{\supset} \dots\dots}$$

L/L^1, L^1/L^2 abelian.

p^{a_1} p^{a_2} ....

$T = L/L^1$ order p^{a_1}, abelian

$$\begin{tikzcd} L \arrow[r, "\text{quotient}", "q"'] \arrow[d, phantom, "p\ \cup"] & T \arrow[d, phantom, "\cup\ p"] \\ q^{-1}(T_0) \arrow[r, "q"] & T_0 \end{tikzcd}$$

contains an index p subgroup T_0

Can refine to

$$L = L_0 \supset L_1 \supset L_2 \dots\dots \supset L_r = 0$$

s.t $|L_i/L_{i+1}| = p$ $L_i/L_{i+1} \cong C_p$

$K/\mathbb{C}$ - deg 2 $\beta \in K \setminus \mathbb{C}$ β deg 2

But any deg 2 polyn in $\mathbb{C}[x]$ factors into linear terms $\sqrt{\Delta}$

$x^2 + bx + c$. Contradiction

□.

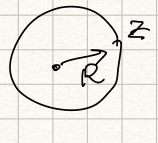
$\overline{\mathbb{C}} = \mathbb{C}$ alg. closed.

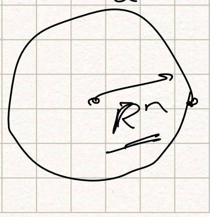
$f(x): \mathbb{C} \longrightarrow \mathbb{C}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left] & \mathbb{C} \\ a \arrow[r] & f(a) \end{tikzcd}$$

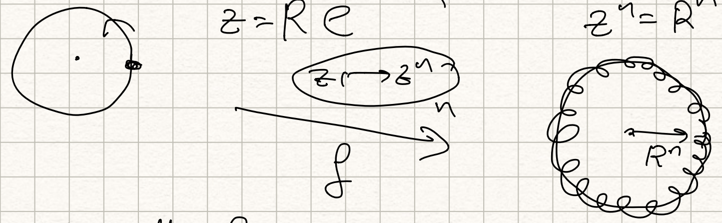
continuous.

$x \longmapsto x^n$
$z \longmapsto z^n$

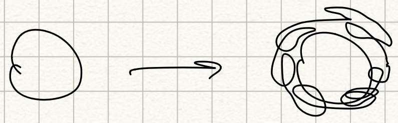


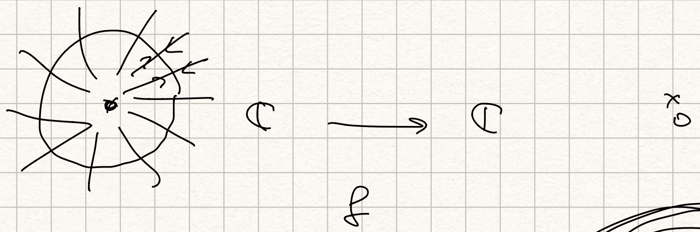


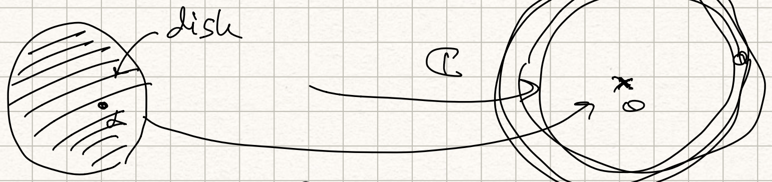
$z = R e^{2\pi i \varphi}$ $\qquad$ $z^n = R^n e^{2\pi i n \varphi}$



winding # for a map from circle to circle







$f(\alpha) = 0$

$x \mapsto x^n$

$x \mapsto f(x) = x^n + a_{n-1} \ldots$ $\qquad$ R

idea of proof (Topology).

D.

$|G| = p^r$ has a subgroup H of index p.

$G/H \cong C_p$ $\qquad$ $[G:H] = p$.

G solvable $\qquad$ if $|G| = p^r$ then

G has nontrivial center $\qquad$ $Z(G) \neq 1$

Conjugation action of G on itself, count orbits of conj. action orbits are conjugacy classes

at least p orbits of size 1

$\{1\}$ all other orbits have order div. by p.

$\overset{\circ}{x}$ $Gx = \mathcal{O}_x$ conj class of x

$$\mathcal{O}_x = \{gxg^{-1} : g \in G\}$$

$$p^r = |G| = \underset{\substack{\uparrow \\ p^{r-m}}}{|\mathcal{O}_x|} \cdot \underset{\substack{\uparrow \\ p^m}}{|\mathrm{Stab}_G(x)|} \qquad m \le r$$

p^{r-m} ↑ either 1 or divisible by p.

$|\mathcal{O}_1| = 1$. have

$|\mathcal{O}_x| = 1 \iff x \in Z(G)$ $\qquad gxg^{-1} = x \ \forall g \in G$.

for conj. action of G on itself, orbits of size 1 are elements of center $Z(G)$.

$|Z(G)| \geq p$.

$$\underset{p^r}{G} \rhd \underset{p^m,\, m>0}{Z(G)} \qquad \overset{\text{solv}}{\underset{p^{r-m}}{\underline{G/Z(G)}}}, \quad \overset{\text{ab, solvable}}{\underline{Z(G)}}$$

& size-wise $|G| = |G/H| \cdot |H|$

G is more complicated than $G/H \times H$.

<u>$\mathbb{C}$ alg. closed</u>

$\underline{V \simeq \mathbb{C}^n}$ or V is a finite dim complex vect. space

$\mathbb{R}^n, F^n$

$V \xrightarrow{L} V$ $\qquad$ $\mathbb{C}^n \xrightarrow{A} \mathbb{C}^n$ $\qquad$ $A = (a_{ij})$

L - lin. operator $\qquad$ $a_{ij} \in \mathbb{C}$

Thm L has an eigenvector, eigenvalue.

$\exists \lambda \in \mathbb{C}, v \in V, v \neq 0$ $\qquad$ $\underline{Lv = \lambda v}$.

$\det(L - x \cdot Id)$ $\qquad$ $\det(A - x \, Id)$

$\underline{\text{pol. of deg } n \text{ in}}$ x

$\uparrow$ variable

$\Rightarrow$ has a root λ

$\uparrow$ compl. coeff

$\underline{\det(A - \lambda Id) = 0}$.

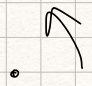
$x = \lambda$

$\det(B) = 0$ $\qquad$ v $\qquad$ $\underline{\underline{Bv = 0}}$

$\underline{A - n \times n, \text{ complex/real.}}$ has an eigenvector/eigenvalue.

$\mathbb{R}$ $\qquad$ rotation matrices.

(no eigenvectors/$\mathbb{R}$)



$\mathbb{R}^2$

$V \xrightarrow{L} V$

$\underline{\text{Classify possible } L}$ $\qquad$ $\underline{\text{find basis}}$ where L has a $\underline{\text{nice form}}$.

best done over $\mathbb{C}$, extends to alg. closed fields $\overline{\mathbb{F}_p}$

(Jordan normal form of a matrix or lin. operator)

Semisimple (diagonalizable) operator or matrix.

$L: V \to V$ diag. if in $\underline{\text{some basis}}$ L has the form

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & 0 \\ 0 & & \lambda_n \end{pmatrix} \qquad A : \mathbb{C}^n \to \mathbb{C}^n$$

$$A \to \underline{BAB^{-1}}$$

"most" complex matrices are diagonalizable

$\underline{\text{not all}}$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \underline{\underline{\lambda=0}} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ not diagonalizable

nilp, eigenvalues $= 0$.

$$\nwarrow$$

$$k$$

$$J_{\lambda,k} = {}^{k}\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix}$$

$\underline{\text{(Jordan normal form}}$ of a matrix)

$\underline{\text{Thm}}$ Any complex $n \times n$ matrix A can be conjugated to a block-diagonal form

$$\begin{pmatrix} \boxed{J_{\lambda_1,k_1}} & & & \\ & \boxed{J_{\lambda_2,k_2}} & & \\ & & \boxed{\ddots} & \\ & & & \boxed{J_{\lambda_n,k_n}} \end{pmatrix} \qquad \begin{pmatrix} \boxed{\lambda_1} & & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{matrix}} & & \\ & & \boxed{\lambda_2} & \\ & & & \ddots \end{pmatrix}$$

any $\overline{F} = F$

2×2 $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ 2 - cases

Classify vect. spaces + operator $\underbrace{V \xrightarrow{L} V}$

$\mathbb{C}[X]$ act on $\underline{\underline{V}}$ by having X act as L.

$\uparrow$

(V, L) is a module $\mathbb{C}$. module/ring.

over $\mathbb{C}[X]$

G-set $G \circlearrowright V$

$\underline{M}$ - ab. group + action of R, compatible

$\underline{R \times M \to M}$.

Ex 1) vect. spaces over F = modules over F

2) ab. groups are modules over $\mathbb{Z}$.

3) vect space + operator = module over $F[x]$

F

4) vect space + action of group G = mod. over $\mathbb{C}[G]$.

$E/\underline{\underline{F}}$ $G : \underline{E} \xrightarrow{g} E$ $g(a) = a$ $a \in F$

example of action on F-vect. space

that is F-linear

F-lin, but not $\underline{\underline{E}}$-lin