lect 23 Discriminants (Rotman p. 95 + Friedman NGT IV p. 9 +)

$$\delta = \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)$$

$\binom{n}{2}$ terms

$\Delta = \delta^2$ discriminant $\delta = \sqrt{\Delta}$

many choices.

2 interpretations

I) $R = F[\alpha_1 \ldots \alpha_n]$ ← polynomials

$\cup$

Sym ← symm functions

Λ_n

II) $G \circlearrowright E = F(\alpha_1 \ldots \alpha_n)$

$\cup$ splitting field extension

F

$f(x) = (x-\alpha_1) \ldots (x-\alpha_n) = x^n - s_1 x^{n-1} + s_2 x^{n-2} - \ldots$

$\ldots + (-1)^n s_n$ s_i - elementary symm. f's.

$\mathrm{Sym} = F[s_1, s_2 \ldots s_n] \subset R$

I $S_n \circlearrowright R$ II $G \circlearrowright E$. $G \circlearrowright X$

G acts on X

$\sigma(\Delta) = \Delta$ $\sigma \in S_n$ or $\sigma \in G$

$\sigma(\delta) = \mathrm{sgn}(\sigma)\,\delta \Rightarrow \sigma\delta = \delta$ (or $\sigma(\sqrt{\Delta}) = \sqrt{\Delta}$)

iff $\sigma \in A_n$.

$F \subset E$ $\sqrt{\Delta} \in F$ sometimes not

$F \subset F(\sqrt{\Delta}) \subset E$

$$\begin{array}{c} E \\ \mid \; G \; S_n \\ F \end{array} \qquad \begin{array}{c} E \\ \mid \\ F(\sqrt{\Delta}) \\ \mid \\ F \end{array} \Big\} \; \underline{\underline{A_n}}$$

Rm $\underline{\underline{G}} \subset A_n$ iff $\underline{\sqrt{\Delta} \in F}$

$$\sigma(\sqrt{\Delta}) = \operatorname{sgn}(\sigma)\sqrt{\Delta}. \qquad \operatorname{sgn}(\sigma) = 1 \;\; \forall \sigma \in G$$

$$G \subset A_n$$

$\deg f = 3$ irr $/F$

E – spl. field

$G \subset S_3$, acts transitively on roots $\Rightarrow$

$G = A_3 \simeq C_3$ or $G = S_3$.

Prop $G = A_3$ iff Δ has a square root in F.

$$[E : F(\sqrt{\Delta})] = 3$$

$$\begin{array}{c} E \\ A_3 \; \mid \; 3 \\ F(\sqrt{\Delta}) \\ 1 \text{ or } 2 \; \mid \\ F \end{array}$$

$\underline{\alpha_1, \alpha_2, \alpha_3} \notin \underline{F(\sqrt{\Delta})}$

$\underline{E = F(\alpha_1, \sqrt{\Delta})}$

any α_i

$E = \underline{F(\alpha_1, \alpha_2, \alpha_3)}$

How to compute Δ as f'n of coeff. of f?

Δ – polyn deg 6 in $\alpha_1, \alpha_2, \alpha_3$ – symm.

polyn in coeff of $f(x)$.

$$\tilde{f}(x) = x^3 + \overset{\alpha_1+\alpha_2+\alpha_3}{a} x^2 + \overset{\alpha_1\alpha_2 \dots}{b} x + \overset{\alpha_1\alpha_2\alpha_3}{c}$$

(1) (2) (3)

$$y = x + \frac{a}{3}$$

$\tilde{f}(y) \qquad y \longmapsto x$

$\underline{\operatorname{char} F = 0}$

$F = \mathbb{Q}$.

$f(x) = x^{3} + px + q$ f's of a, b, c

Thm (1) $\tilde{f}(x)$ and $f(x)$ have the same discriminant. $\Delta(\tilde{f}) = \Delta(f)$

(2) $\boxed{\Delta = -4p^3 - 27q^2}$

6 2

2^2 3^3

$\deg p = 2$

$\deg q = 3$

p^3, q^2

Rotman D instead of Δ.

$f(x) = x^3 + px + q$ $\alpha_1, \alpha_2, \alpha_3$

$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$.

$= x^3 - \underbrace{(\alpha_1 + \alpha_2 + \alpha_3)}_{0} x^2 \ldots$

$$\boxed{\begin{aligned} &\alpha_1 + \alpha_2 + \alpha_3 = 0 \\ &\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = p \\ &\alpha_1\alpha_2\alpha_3 = -q \end{aligned}}$$

$\left(\prod_{i<j}(\alpha_i - \alpha_j)\right)^2$

$f'(x) = (x - \alpha_2)(x - \alpha_3) + (x - \alpha_1)(x - \alpha_3) + (x - \alpha_1)(x - \alpha_2)$.

$f'(\alpha_1) = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$

$f'(\alpha_2) = (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)$

$f'(\alpha_3) = (\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)$

$f'(\alpha_1) f'(\alpha_2) f'(\alpha_3) = -\Delta(f)$.

$f'(x) = 3x^2 + p$

$-\Delta(f) = (3\alpha_1^2 + p)(3\alpha_2^2 + p)(3\alpha_3^2 + p) =$

$= 27\alpha_1^2\alpha_2^2\alpha_3^2 + 9p(\alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_3^2) + 3p^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + p^3$

$(\alpha_1\alpha_2\alpha_3)^2 = (-q)^2 = q^2.$ manipulate symm f's

$$0 = \underline{(\alpha_1+\alpha_2+\alpha_3)^2} = \underset{\substack{\| \\ -2p.}}{\underline{\alpha_1^2+\alpha_2^2+\alpha_3^2}} + 2\underbrace{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)}_{}$$

$$p^2 = (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^2 = \underbrace{\alpha_1^2\alpha_2^2+\alpha_1^2\alpha_3^2+\alpha_2^2\alpha_3^2}_{\substack{\| \\ p^2}} + 2\underbrace{(\alpha_1\alpha_2\alpha_3)\overset{\substack{0 \\ \|}}{\underline{(\alpha_1+\alpha_2+\alpha_3)}}}_{}$$

$$-\Delta(f) = 27q^2 + 9p\cdot p^2 + 3p^2(-2p) + p^3 = 27q^2+4p^3$$

$$\boxed{\Delta = -4p^3 - 27q^2}$$

Ex 1) $f = x^3 - 2$, $G = S_3$, $\Delta = -27(-2)^2 =$

$p=0, q=-2$ $= -27\cdot 4 = -3(6^2)$

$\sqrt{\Delta} \notin \mathbb{Q}.$

2) $f = x^3 - 3x + 1$ irr$/\mathbb{Q}$ root test

$p=-3$ $q=1$ $\Delta = -4(-3)^3 - 27 = 4\cdot 27 - 27 = 3\cdot 3^3 =$

$= 9^2$ $\sqrt{\Delta} = \pm 9 \in \mathbb{Q}.$

$\Rightarrow G = C_3$ - cyclic $A_3 \simeq C_3.$ rare, usually

$$F \subset F(\sqrt{\Delta}) \underset{3}{\overset{\leftarrow C_3.}{\subset}} E$$

$\sqrt{\Delta} \notin \mathbb{Q}.$

$f(x) = x^3 + px + q.$

$\alpha_1, \alpha_2, \alpha_3$ $\quad \alpha_1+\alpha_2+\alpha_3 = 0.$

$p = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$

$q = -\alpha_1\alpha_2\alpha_3$

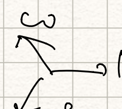
Look for combination of roots to realize E as a radical ext of deg. 3

$Gal(E/F(\sqrt{\Delta})) \simeq A_3 \subset S_3$ $\qquad \sigma = (123)$

add $\sqrt[3]{1} = \omega$

$1+\omega+\omega^2=0$

$Z = \alpha_1 + \omega\alpha_2 + \omega^2\alpha_3$



$F \rightsquigarrow F(\omega)$

$\sigma(Z) = \sigma(\alpha_1)+\sigma(\omega\alpha_2)+\sigma(\omega^2\alpha_3)$

$= \sigma(\alpha_1)+\omega\sigma(\alpha_2)+\omega^2\sigma(\alpha_3)$

$\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$ (over $\mathbb{C}$)

$$\begin{tikzcd} \alpha_1 \arrow[r, "\sigma"'] & \alpha_2 \arrow[r, "\sigma"'] & \alpha_3 \arrow[ll, bend right, "\sigma"'] \end{tikzcd}$$

$E \rightsquigarrow E(\omega) = E'$

$F(\sqrt{\Delta}) \rightsquigarrow F(\sqrt{\Delta}, \omega)$

$F \rightsquigarrow F(\omega) = F$

$\sigma(\omega) = \omega$

$= \alpha_2 + \omega\alpha_3 + \omega^2\alpha_1 =$

$= \omega^2(\alpha_1 + \omega\alpha_2 + \omega^2\alpha_3) =$

$= \omega^2 Z.$

$$\boxed{\sigma(Z) = \omega^2 Z}$$

Z is an eigenvector of operator σ of eigenvalue ω^2

σ is F'-linear

$\sigma(\sqrt{\Delta}) = \sqrt{\Delta}$

$\sigma(\omega) = \omega$

σ is identity on F'

$$\begin{tikzcd} E \arrow[r, squiggly] & E(\omega) = E' \\ F(\sqrt{\Delta}) \arrow[u, phantom, "\cup"] \arrow[r, squiggly] & F(\sqrt{\Delta}, \omega) = F' \arrow[u, phantom, "\cup\ 3"] \\ F \arrow[u, phantom, "\cup"] \arrow[r] & F(\omega) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

E' is a 3d vect space $/F'$

$\sigma^3 = 1$ $\qquad$ diagonalize σ. $\qquad z = \alpha_1 + \omega\alpha_2 + \omega^2\alpha_3$

E' $\quad 1, \omega, \omega^2$ $\qquad Z' = \alpha_1 + \omega^2\alpha_2 + \omega\alpha_3$

$\cup \qquad \uparrow \quad \uparrow \quad \uparrow$

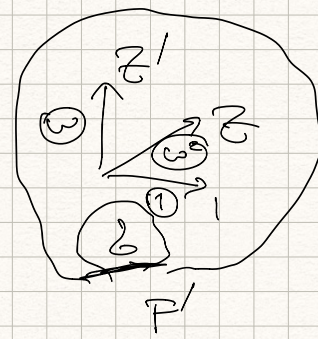
$F' \ni 1 \quad Z' \quad Z$ $\qquad \sigma(Z') = \omega Z'$

$\sigma(1) = 1$

$z+z' = 2\alpha_1-\alpha_2-\alpha_3 = \underline{3\alpha_1}$

$\alpha_1 = \frac{1}{3}(z+z'),\ \alpha_2 = \frac{1}{3}(\omega^2 z + \omega z')$

$\alpha_3 = -\alpha_1-\alpha_2.$



$\sigma(z\cdot z') = \underbrace{\omega^2\omega}_{1}\, zz' = zz'$

Ex $\boxed{z\cdot z' = -3p}$ $\quad p\neq 0 \iff z, z'\neq 0.$

Computation to write z^3

(see Friedman p.48)

$\sigma(z^3) = \omega^6 z^3 = z^3$

$\Downarrow$

$z^3 \in F' = F(\sqrt{\Delta},\omega) \ni q, p$

$$z^3 = -9q + \frac{9}{2}(\overbrace{\omega+\omega^2}^{-1})q + \frac{3}{2}(\underbrace{\omega-\omega^2}_{\sqrt{-3}})\sqrt{\Delta} \in F(\sqrt{\Delta},\omega).$$

$$z^3 = -\frac{27}{2}q + \frac{3}{2}\sqrt{-3\Delta}$$

$$\boxed{z = \sqrt[3]{-\frac{27q}{2} + \frac{3}{2}\sqrt{-3\Delta}}}$$

$$z' = (-3p)z^{-1} = \sqrt{-\frac{27q}{2} - \frac{3}{2}\sqrt{-3\Delta}}$$

$$\frac{z+z'}{3},\quad \frac{\omega^2 z+\omega z'}{3},\quad \frac{\omega z+\omega^2 z'}{3}.$$

$$F \rightsquigarrow F(\sqrt{\Delta},\omega) \rightsquigarrow F(z,\sqrt{\Delta},\omega) \ni \alpha_1,\alpha_2,\alpha_3$$

$\forall$ irr. deg 3 polyn. is solvable in radicals!

$G\subset S_3$ - solvable.

Direction to explore → group action on vect. spaces

$F(\sqrt{\Delta}, z)$

$\mathbb{R}, \mathbb{C}, \quad \underline{\underline{F}}$

$F - F(\sqrt{\Delta}) - E''$

$\omega \qquad \omega \qquad A_3$

this extension is radical.

$0 = \alpha_1 + \alpha_2 + \alpha_3$

$z = \alpha_1 + \omega\alpha_2 + \omega^2\alpha_3$

$z' = \alpha_1 + \omega^2\alpha_2 + \omega\alpha_3$

$$\underline{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ z \\ z' \end{pmatrix}$$

↑

Vandermonde matrix/determinant

$$A = \begin{pmatrix} 1 & 1 & & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & & & \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{pmatrix}$$

det A - polyn in $\alpha_1 \dots \alpha_n$.

$1, \alpha_i, \alpha_j^2, \alpha_k^3 \dots \alpha_e^{n-1}$

$0 + 1 + 2 + \dots \qquad n-1$

each term has deg

$1 + 2 + \dots + n-1 = \binom{n}{2} = \frac{n(n-1)}{2}$

$\begin{pmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{pmatrix} = \underline{\alpha_2 - \alpha_1}$

$\underline{\alpha_i - \alpha_j} \mid \underline{\det A}$

$A \mid_{\alpha_i = \alpha_j}$

$$0 \xleftarrow{\det} \begin{array}{cc} i & j \\ \left(\begin{matrix} 1 & 1 \\ \alpha_i & \alpha_i \\ \alpha_i^2 & \alpha_i^2 \\ \vdots & \vdots \\ \alpha_i^{n-1} & \alpha_i^{n-1} \end{matrix} \right) \end{array}$$

$\left| \begin{matrix} \alpha_i^k - \alpha_j^k \\ \vdots \\ \vdots \end{matrix} \right| \qquad (\alpha_i - \alpha_j)$

$$\left(\quad f(x,y) \quad \right) \qquad f(x,y). \qquad f(x,x)=0.$$

$$\Rightarrow x-y \mid f(x,y).$$

Ex

$\Rightarrow \quad \alpha_i - \alpha_j \mid \det A \qquad i<j$

coprime diff $i,j \Rightarrow \prod_{i<j} (\alpha_i - \alpha_j) \mid \det A$

$\frac{n(n-1)}{2}$ $\qquad$ deg $\binom{n}{2}$

Thm $\det A = (-1)^{\binom{n}{2}} \prod_{i<j} (\alpha_i - \alpha_j) = \prod_{1\le i<j\le n} (\alpha_j - \alpha_i)$

$\delta \qquad \Delta = \delta^2$

deg 4 $\qquad f(x) = x^4 + ax^3 + \dots$

4 roots $\alpha_1 \dots \alpha_4 \qquad G \in S_4.$

$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$ $\quad$ not symm

$\beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$ $\quad S_4$

$\beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$

$$E = F(\alpha_1 \dots \alpha_4)$$
$$\Big|\ 4$$
$$E^H = F(\beta_1, \beta_2, \beta_3)$$
$$\Big|$$
$$F(\sqrt{\Delta})$$
$$\Big|$$
$$F$$

$$H = \begin{Bmatrix} (12)(34) \\ (13)(24) \\ (14)(23) \\ id \end{Bmatrix} \cong C_2 \times C_2 \subset S_4.$$

$$(x-\beta_1)(x-\beta_2)(x-\beta_3) \in F[x].$$

first β's then α's.

$$\{1\} \subset H \subset A_4 \subset S_4$$

$$E \leftarrow E^{M} \leftarrow F(r_\Delta) \leftarrow F$$