Discriminants

(Rotman p. 95+, Friedman NGT IV, p. 9+)

$\delta = \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)$ $\Delta = \delta^2$ discriminant $\delta = \sqrt{\Delta}$

2 interpretations

$f(x) = (x-\alpha_1)\dots(x-\alpha_n) = x^n - s_1 x^{n-1} + s_2 x^{n-2} \dots + (-1)^n s_n$

$R = F[\alpha_1 \dots \alpha_n]$ polyn
$\cup$
Sym
$\uparrow$
symmetric f's

$E = F(\alpha_1 \dots \alpha_n)$ splitting field extension
$\cup$
F

s_i - elementary symmetric f's of α's $\mathrm{Sym} = F[s_1 \dots s_n]$

S_n acts on R, permutes α_i's

$G = \mathrm{Gal}(E/F) \subset S_n$ acts on E, permutes α_i's
$\uparrow$ S_n or smaller

$\sigma\Delta = \Delta \quad \forall \sigma \in S_n$

$\sigma(\delta) = \mathrm{sgn}(\sigma)(\delta)$ $\Rightarrow$ $\sigma\delta = \delta$ (or $\sigma(\sqrt{\Delta}) = \sqrt{\Delta}$) iff $\sigma \in A_n$

Express Δ as f'n of coefficients of $f(x)$, take square root

$F \subset F(\sqrt{\Delta}) \subset E$

<u>Thm</u> For F, E, α's as above (& no multiple roots, $\alpha_i \ne \alpha_j$)

$G = \mathrm{Gal}(E/F) \subset S_n$. Then $G \subset A_n$ (alternating group) iff $\sqrt{\Delta} \in F$.

$\sigma(\sqrt{\Delta}) = \mathrm{sgn}(\sigma)\sqrt{\Delta}$.   if $\sqrt{\Delta} \in F$ $\mathrm{sgn}(\sigma) = 1 \quad \forall \sigma \in G$.

(see Friedman NGT IV prop 10.2 or Rotman).

Suppose f has degree 3, irreducible $/F$. E-splitting field (char $F=0$)

$G \subset S_3$, G acts transitively on roots $\Rightarrow G = A_3$ or $G = S_3$.
$A_3 \cong C_3$

Prop $G = A_3$ iff Δ (discriminant) has a square root in F. $\square$

(most of the time Δ is not a square in F and $G = S_3$. Then

$\mathrm{Gal}(E/F) = S_3$

$\mathrm{Gal}(E/F(\sqrt{\Delta})) = A_3$

$\mathrm{Gal}(F(\sqrt{\Delta})/F) = C_2$

$$E \overset{3}{—} F(\sqrt{\Delta}) \overset{2}{—} F$$

$E = F(\alpha_i, \sqrt{\Delta})$
↑
any of the 3 roots

Since $[E : F(\sqrt{\Delta})] = 3$, $\alpha_i \notin \sqrt{\Delta}$.

$1 \subset A_3 \subset S_3$ $\quad S_3/C_3 \simeq C_2$, $C_3 \cong A_3$ normal

$E \leftarrow F(\sqrt{\Delta}) \leftarrow F$

$\tilde{f}(x) = x^3 + ax^2 + bx + c$ $\quad y = x + \frac{a}{3}$ $\quad$ char $F \neq 3$. $\quad a, b, c \in F$

relabel y into x

$f(x) = x^3 + px + q$, $\quad p, q \in F$

Thm (1) $\tilde{f}(x)$ and associated reduced polynomial $f(x)$ have the same discriminant

(2) Discriminant $\Delta = -4p^3 - 27q^2$

Proof (1) is easy (nothing happens to $\alpha_i - \alpha_j$ upon shift). or see Rotman Thm 100 p. 97

(2) Follow Friedman or see lect 21 p 4-6.

		discriminant			
Notation discrepancy:	Friedman:	Δ	p, q	in	$x^3 + px + q$
	Rotman	D	q, r	in	$x^3 + qx + r$

follow Friedman.

$f(x) \leftrightarrow \tilde{f}(x)$.

$$\boxed{\Delta = \Delta(f) = -4p^3 - 27q^2}$$

Δ - degree 6 in $\alpha_1, \alpha_2, \alpha_3$

p - degree 2

q - degree 3

(and $\alpha_1 + \alpha_2 + \alpha_3 = 0$).

only way to get to degree 6 is via p^3 or q^2.

$4 = 2^2$, $27 = 3^3$ $\quad \Delta = -2^2 p^3 - 3^3 q^2$

if you specialize $p=0$ or $q=0$ & write down Δ, these coefficients are easy to see/understand

Examples 1) $f = x^3 - 2$, $\quad \Delta = -27(-2)^2 = -27 \cdot 4 = -3(6^2)$

$p=0, q=-2$ $\qquad \sqrt{\Delta} \notin \mathbb{Q}$

2) $f = x^3 - 3x + 1$ $\quad$ irreducible/$\mathbb{Q}$ by rational roots test

$p=-3, q=-1$

$$\Delta = -4(-3)^3 - 27(-1)^2 = 4 \cdot 27 - 27 = 3 \cdot 27 = 9^2$$

$$\Rightarrow \sqrt{\Delta} = \pm 9 \in F = \mathbb{Q}.$$

$\Rightarrow G = C_3$ cyclic $(C_3 \cong A_3)$.

Most of the time $\sqrt{\Delta} \notin \mathbb{Q}$ & $G = S_3$.

Vandermonde determinant (see lect 21 notes, p. 6).

$f(x) = x^3 + px + q$ irreducible $\qquad F \subset F(\sqrt{\Delta}) \subset E \quad \text{char } F = 0.$

$\alpha_1, \alpha_2, \alpha_3$ roots $\qquad \alpha_1 + \alpha_2 + \alpha_3 = 0. \qquad p = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$

$q = \alpha_1 \alpha_2 \alpha_3$

Try to realize E/F as a radical extension

Look for elements that lie in $F(\sqrt{\Delta})$ when cubed

$Gal(E/F(\sqrt{\Delta})) = A_3 \subset S_3 \qquad \sigma = (123)$ label roots correspondingly.

add third roots of unity $\omega, \omega^2, \omega^3 = 1 \qquad \omega = \sqrt[3]{1} = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$

$\uparrow$ over $\mathbb{C}$.

$$Z = \alpha_1 + \omega\alpha_2 + \omega^2\alpha_3$$

$$\sigma(Z) = \sigma(\alpha_1) + \sigma(\omega\alpha_2) + \sigma(\omega^2\alpha_3) = \alpha_2 + \omega\alpha_3 + \omega^2\alpha_1 = \omega^2(\alpha_1 + \omega\alpha_2 + \omega^2\alpha_3) =$$
$$= \omega^2 Z$$

Z is an eigenvector of σ of eigenvalue ω^2. $\quad \left(\sigma^3 = 1 \to \text{possible eigenvalues are } 1, \omega, \omega^2 \right)$

$\Rightarrow Z^3$ is an eigenvector of σ of eigenvalue $1 \Rightarrow$

expect that $Z^3 \in F(\sqrt{\Delta}, \omega)$..

$$\begin{array}{ccc} E' & = & E(\omega) \\ \scriptstyle 3 \cup & & \cup \scriptstyle 3 \\ F' & = & F(\sqrt{\Delta}, \omega) \end{array}$$

companion of Z:

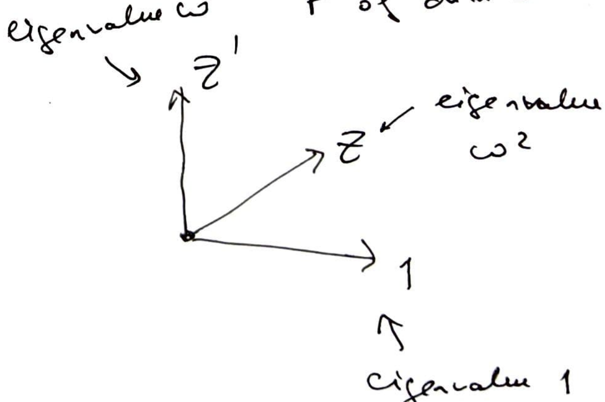
$$Z' = \alpha_1 + \omega^2\alpha_2 + \omega\alpha_3$$

$$\sigma(Z') = \omega Z'$$

σ acts on E'

$\uparrow$

vect. space over F' of dim 3



Coefficients in F'!

Vandermonde matrix, for roots of unity

$$0 = \alpha_1 + \alpha_2 + \alpha_3$$
$$z = \alpha_1 + \omega\alpha_2 + \omega^2\alpha_3$$
$$z' = \alpha_1 + \omega^2\alpha_2 + \omega\alpha_3$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ z \\ z' \end{pmatrix}$$

$1 + \omega + \omega^2 = 0$

$$z + z' = 2\alpha_1 - \alpha_2 - \alpha_3 = 3\alpha_1 \qquad \alpha_1 = \frac{1}{3}(z + z')$$

$$\alpha_2 = \frac{1}{3}(\omega^2 z + \omega z')$$

$$\alpha_3 = -\alpha_1 - \alpha_2$$

<u>Exercise</u> $z \cdot z' = -3p.$

For explicit formula for z^3, see computation in Friedman, p. 48.

$$z^3 = -9q + \frac{9}{2}(\omega + \omega^2)q + \frac{3}{2}(\omega - \omega^2)\sqrt{\Delta} \in F(\sqrt{\Delta}, \omega).$$

$$\omega + \omega^2 = -1$$
$$\omega - \omega^2 = \sqrt{-3}$$

$$z^3 = -9q - \frac{9}{2}q + \frac{3}{2}\sqrt{-3\Delta} = -\frac{27}{2}q + \frac{3}{2}\sqrt{-3\Delta}$$

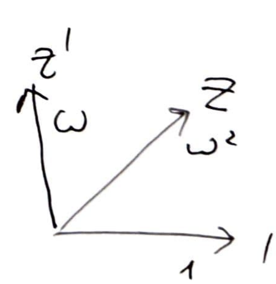
$$z = \sqrt[3]{-\frac{27q}{2} + \frac{3}{2}\sqrt{-3\Delta}}, \quad z' = (-3p)z^{-1} = \sqrt[3]{-\frac{27q}{2} - \frac{3}{2}\sqrt{-3\Delta}}$$

if $z \neq 0$; $z = 0$ - easy case. $(p = 0)$

Roots are

$$\frac{z + z'}{3}, \quad \frac{\omega^2 z + \omega z'}{3}, \quad \frac{\omega z + \omega^2 z'}{3}.$$

Had to add ω (3rd root of unity), diagonalize $\sigma = (123)$.



Degree 4 polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$

$\alpha_1, \alpha_2, \alpha_3, \alpha_4$ roots $\qquad G \subset S_4$

$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$

$\beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$

$\beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$.

$$\begin{array}{c} E \\ |\ 24 \\ F \end{array} \qquad \begin{array}{c} E \\ 4 \\ E^H \\ |\ 6 \\ F \end{array}$$

$S_4 \xrightarrow{\varphi} S_3$

Conj. class $\quad (12)(34)$, $(13)(24)$, $(14)(23)$ $+ id = H \subset S_4$

$H = \ker\varphi$. normal

G permutes $\beta_1, \beta_2, \beta_3$.

H preserves $\beta_1, \beta_2, \beta_3$.

$$\begin{array}{c} E = F(\alpha_i) \\ | \\ E^H = F(\beta_j) \\ | \\ F \end{array}$$

$(x-\beta_1)(x-\beta_2)(x-\beta_3) \in F[x]$

Express β_j as symm f's in coeff of f.

Solve cubic. (resolvent cubic)

then solve deg ≤ 4 extension E/E^H

$C_2 \times C_2$

$\wr\wr$

$H \subset S_4 \qquad S_4/H \simeq S_3$.

$$\begin{array}{c} E \\ | \\ E^H \\ |\ A_3 \\ F(\sqrt{\Delta}) \\ |\ C_2 \\ F \end{array}$$

$C_2 \times C_2$ Galois group, quadratic extensions

discriminant of resolvent cubic

$\{1\} \subset H \subset A_4 \subset S_4$

$E \leftarrow E^H \leftarrow F(\sqrt{\Delta}) \leftarrow F$

See Rotman or Friedman for details.