Let $F_0 = \mathbb{Q}$, start here. Keep adding $\sqrt[n]{c}$, various $c$ and $n$.

(Rodman, p. 71)

Def: $B/F$ is a pure extension of type $m$ if $B = F(t)$, $t^m \in F$ some $m$.

Let $B = \mathbb{Q}_c$, $c \in B$. $cB$ is radical down if each $B/K/B_i$ is a pure extension.
Call $B/F$ radical extension.

Wish to show $Gal(B/F)$ is not too complicated. (Solvable group).

If $B = F(t)$ pure of type $m$ and $F$ contains a cube of unity,
then $B/F$ is a splitting field of $x^m - t^m$, $x - c$. $c = \sqrt[m]{t}$

Def: $f \in F[x]$, $f$ is solvable by radicals on $F$ if $f$ a radical extension which contains a splitting field $F$ of $f/F$.

Let $g(x) \in F[x]$ be solvable by radicals on $F$, then $F = F_0$ and $f \in F/F$ be its splitting field. Then $G = Gal(E/F)$ is a solvable group.
Solvable groups (Primer, appendix B)

\[ G \rightarrow G^{(1)} = [G, G] \text{ normal subgroup} \]

\[ G^{(2)} = [G^{(1)}, G^{(1)}] \]

\[ G^{(n)} = [G^{(n-1)}, G^{(n-1)}] \]

Example: Invertible upper triangular matrices \( n \times n \)

\( S_n, A_n - \) not solvable. \( A_n = [A_n, A_n] \)

smallest non-abelian simple group: \( A_5 \) of order 60

other examples of finite simple groups

- \( GL(n, \mathbb{F}_p) \)
- \( PGL(n, \mathbb{F}_p) \)

Prop. If \( K \triangleleft G \) normal then

\( G \) solvable \( \Rightarrow K \) solvable, \( G/K \) solvable.

\[ F_0 \subset F_1 \subset F_2 \subset F_3 \ldots \subset F_n \]

\( G = \text{Gal}(F_n/F_0) \) and \( \varphi_0 = \text{Gal}(F_n/F_0) \)

\( \downarrow \varphi_0 \)

\( \text{Gal}(F_1/F_0) \)

\( \uparrow \varphi_1 \)

\( \text{Gal}(F_2/F_0) \)

\( \text{Gal}(F_3/F_2) \)

\(...\)

\( G \supset G_1 \supset G_2 \ldots \supset G_n = \{1\} \)

\( \text{ker } \varphi_1 = G_2 = \text{Gal}(F_n/F_2) \)

\( \downarrow \varphi_2 \)

\( \text{Gal}(F_3/F_2) \)

\( \ldots \)

\( G_{n-1} \text{ is normal} \)
There is a degree 5 polynomial \( p(x) \) not solvable by radicals.

**Proof**  
\( p(x) = x^5 - 4x + 2 \)

Eisenstein crt \( \Rightarrow \) irreducible \( \mathbb{Q} \)

Let \( E/Q \) be splitting field, \( E \subseteq \mathbb{C} \) (use and \( C \) is alg. closed).

\[ G = \text{Gal}(E/Q) \quad \text{d - real alg \( p(x) \)} \]

\[ E > Q(\alpha) \quad 5 \in \mathbb{Q} \quad [E:Q(\alpha)] = [E:Q] [Q(\alpha):Q] = 5 \quad [E:Q(\alpha)] \]

\[ 5 \mid 161 = [E:Q] \]

\( p(x) = 5x^5 - 4x + 2 \) and pts \( x^5 = \frac{4}{5} \), \( x = \pm 0.946 \)

3 real roots, \( \sim -1.52, 0.5, 1.24 \) \( \Rightarrow \) complex orj in \( G \) restricts to a transposition \( b \).

\( G \) contains elt of order 5 \( \Rightarrow \) contains a cycle.

**Lemma** (Rotman, Ch. 35, p.121) \( d \in S_5 \) a 5-cycle, \( a \) a transposition, \( d = (12345) \) \( \Rightarrow \)

\[ \langle d, a \rangle = S_5 \]

**Proof** \( H = \langle d, a \rangle \subseteq S_5 \) subgroup. Can assume \( d = (12345) \)

\[ c = (1i) \quad d^c = (51) \]

\[ 2^k (1i) = (i) \quad 5 \rightarrow j, \quad j = d^4 (1) \]

\[ (1i)(ij) = (1ij) \quad \text{order 3}. \]

\[ 2, 3, 5 \mid |H| \Rightarrow \text{30} \mid |H| \]

\[ |S_5| = 120 \]

\[ 5 \rightarrow j \]
$|H| = 30, 60, 120$

$H = S_5$

no such subgroups in $S_5$

$(H \triangleleft A_5)$ since $H$ enters a transposition (odd permutation)

$K \leq S_5 \quad [S_5: K] = 4 \implies S_5 \rightarrow S_4$

homomorphism

$A_5 \subset S_5 \rightarrow S_4$

order 24.

$A_5$ has no $y$ such that $y^2 = 1$

$|L| = 1$ but 1 not in $S_5$

$|L| \geq \frac{|S_5|}{2} = 3$. But $A_5$ is simple. (no normal subgroups)