Lie groups

Homework #4, due Wednesday, April 4.

This homework covers the material of Chapter 2 of A.Kleshchev's book Linear and Projective Representations of Symmetric Groups.

1. What is the dimension of the algebra generated by the Jucys-Murphy elements $L_1, \ldots, L_4$ of $\mathbb{C}[S_4]$? Avoid an explicit computation.

2. List all possible eigenvalues for actions of (a) L_2, (b) L_3, (c) L_4 on representations of S_5. Explain your answer.

3. For which dimensions n is the algebra generated by the Jucys-Murphy elements of S_n coincides with the center of $\mathbb{C}[S_n]$?

4. Explain why the GZ-basis (Gelfand-Zeitlin basis) of an irreducible representation of S_n is orthogonal with respect to the unique (up to scaling) symmetric bilinear S_n-invariant form on the representation. (Also see discussion on page 9 of Kleshchev's book.)

5. Choose any three GZ basis vectors in the irreducible representation $V_{(3,2)}$ of S_5 and write down the eigenvalues of the action of $L_1, L_2, \ldots, L_5$ on these vectors. To avoid possible confusion with the notation for Jucys-Murphy elements, here we write irreducible representations as V_λ rather than L_λ.

6. Which of the irreducible representations $L(a,b)$ of the algebra $\mathcal{H}_2$ described in Section 2.2 remain irreducible upon restriction to the subalgebra (a) $\mathbb{C}[x,y]$, (b) $\mathbb{C}[s]/(s^2-1)$?

7. In class we've constructed idempotents $e_{i,k}$ as polynomials in L_k, for $1-k \leq i \leq k-1$ using Lagrange's interpolation. Write down these idempotents when $n=3$ and $k=1,2,3$ and use their products to give formulas for the GZ basis in the corresponding maximal commutative semisimple subalgebra of S_3.

These idempotents are not in the book, but there's a discussion of them in `https://mathoverflow.net/questions/167792/minimal-idempotents-for-the-group-algebra-of-the-symm`