Lie groups and representations, Fall 2009

Homework #8, due Monday, December 7.

1. Check that the Casimir element $c = \sum x_i y_i$ of the universal enveloping algebra of a simple Lie algebra $L$ does not depend on the choice of basis $\{x_1, \ldots, x_n\}$ of $L$.

2. Let $V$ be a nontrivial irreducible representation of a simple Lie algebra $L$. Explain why the bilinear form $B_V(x, y) = \text{Tr}_V(xy)$ is nonzero, nondegenerate, and proportional to the Killing form, and Casimir elements $c$ and $c_V$ are proportional.

3. Compute multiplicities of irreducible representations in $S^2(V_n)$ and $\Lambda^2(V_n)$. How does $S^n(V_2)$ decompose into irreducibles? Here $V_n$ is the $(n + 1)$-dimensional irreducible $sl(2)$ representation.

4. Let $A$ be the associative algebra of upper-triangular complex $n \times n$-matrices. It acts on the space $V$ of column vectors. Is this representation of $A$ irreducible? Find the commutant $A'$, the second commutant $A''$, and verify that $A \subset A''$ is a proper inclusion for $n > 1$.

5. Let $W$ be a real vector space. Prove that $S^n(W)$ is spanned by the set $\{w \otimes w | w \in W\}$.

6. Compute the action of the normalized Casimir operator $c = ef + fe + h^2$ on each irreducible representation $V_n$. Given a finite-dimensional representation $V$ of $sl(2)$, how can you use $c$ to decompose $V$ into isotypical components?

7. Let $L$ be the 2-dimensional Lie algebra with basis $\{x, y\}$ and relation $[x, y] = y$. Prove that the center of $UL$ consists of constants only.