Lie groups and representations, Fall 2009

Homework #8, due Monday, December 7.

1. Check that the Casimir element $c = \sum x_i y_i$ of the universal enveloping algebra of a simple Lie algebra L does not depend on the choice of basis $\{x_1, \ldots, x_n\}$ of L.

2. Let V be a nontrivial irreducible representation of a simple Lie algebra L. Explain why the bilinear form $B_V(x, y) = \mathrm{Tr}_V(xy)$ is nonzero, nondegenerate, and proportional to the Killing form, and Casimir elements c and c_V are proportional.

3. Compute multiplicities of irreducible representations in $S^2(V_n)$ and $\Lambda^2(V_n)$. How does $S^n(V_2)$ decompose into irreducibles? Here V_n is the $(n+1)$-dimensional irreducible $sl(2)$ representation.

4. Let A be the associative algebra of upper-triangular complex $n \times n$-matrices. It acts on the space V of column vectors. Is this representation of A irreducible? Find the commutant A', the second commutant A'', and verify that $A \subset A''$ is a proper inclusion for $n > 1$.

5. Let W be a real vector space. Prove that $S^n(W)$ is spanned by the set $\{w^{\otimes n} | w \in W\}$.

6. Compute the action of the normalized Casimir operator $c = ef + fe + \frac{h^2}{2}$ on each irreducible representation V_n. Given a finite-dimensional representation V of $sl(2)$, how can you use c to decompose V into isotypical components?

7. Let L be the 2-dimensional Lie algebra with basis $\{x, y\}$ and relation $[x, y] = y$. Prove that the center of UL consists of constants only.