Lie groups and representations, Fall 2009

Homework #6, due Monday, November 9.

1. (a) Show that any continuous homomorphism from $(\mathbb{R}, +)$ to a Lie group G is smooth.

(b) Show that any continuous homomorphism of Lie groups $G \longrightarrow H$ is smooth.

2. For which of the following Lie algebras the adjoint representation is faithful:

- abelian Lie algebras,
- the unique 2-dimensional nonabelian Lie algebra,
- $\mathfrak{gl}(n, \mathbb{R})$,
- $\mathfrak{sl}(n, \mathbb{R})$,
- $\mathfrak{su}(2)$,
- the Lie algebra of $n \times n$ strictly upper-triangular matrices,
- the Lie algebra of $n \times n$ upper-triangular matrices?

3. Let L be the Lie algebra of a connected Lie group G and $I \subset L$ an ideal. What can you say about the connected subgroup H of G that corresponds to I. Is it normal? Is it a closed subgroup? What if G is simply-connected?

4. Show that $GL(n, \mathbb{C})$ is diffeomorphic to $U(n) \times \mathbb{R}^{n^2}$.

5. Show that the adjoint representation of $GL(n, \mathbb{R})$ is given by $\mathrm{Ad}_g(X) = gXg^{-1}$, where $g \in GL(n, \mathbb{R})$ and $X \in \mathfrak{gl}(n, \mathbb{R})$.