Lie groups and representations, Fall 2009

Homework #4, due Monday, October 19.

1. Classify elements of the group $O(1,1)$ and show that it has four connected components. Is $O(1,1)$ isomorphic to $\mathbb{R} \times \mathbb{Z}/2 \times \mathbb{Z}/2$?

2. Finish the proof outlined in class that $SO(2,\mathbb{C}) \cong \mathbb{C}^*$. Extend this isomorphism to an explicit description of $O(2,\mathbb{C})$.

3. Show that the log map restricts to a bijection from a neighbourhood of I in $O(p,q)$ to an neighbourhood of 0 in the space

$$\mathfrak{o}(p,q) := \{A \in \mathfrak{gl}(p+q,\mathbb{R}) | AI_{p,q} + I_{p,q}A^T = 0\},$$

where $I_{p,q}$ is the diagonal matrix with p ones followed by q minus ones. Conclude that $\mathfrak{o}(p,q)$ is the tangent space to the identity of $SO(p,q)$ and a Lie algebra. Describe a basis of $\mathfrak{o}(p,q)$ and compute its dimension.

4. Let A be an associative algebra over a field k. A k-linear endomorphism $d : A \longrightarrow A$ is called a derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. Check that the commutator of derivations is a derivation. Find a basis in the Lie algebra of derivations of the polynomial algebra $k[x]$ and compute the Lie bracket in this basis.

5. (a) Check directly that $\mathfrak{o}(n,\mathbb{R}), \mathfrak{o}(p,q), \mathfrak{u}(n)$ are closed under the commutator bracket, so that they are Lie subalgebras of $\mathfrak{gl}(n,\mathbb{R})$ or $\mathfrak{gl}(n,\mathbb{C})$, respectively.

(b) Check that the space of divergence-free vector fields is a Lie subalgebra of $\mathrm{Vect}(\mathbb{R}^n)$. A vector field $\xi = \sum a_i(x)\frac{\partial}{\partial x_i}$ is divergence-free if

$$0 = \mathrm{div}\xi := \sum \frac{\partial a_i(x)}{\partial x_i}.$$

6. Suppose that vector fields ξ, ξ' are proportional: $\xi' = f(x)\xi$ for some function $f(x)$. Does this imply that they commute?

7. Show that the exponential map $\exp : \mathfrak{gl}(n,\mathbb{R}) \longrightarrow GL^+(n,\mathbb{R})$ from the space of real matrices to the connected component of the identity in $GL(n,\mathbb{R})$ is not surjective.