Algebraic topology, Spring 2014

Homework 2, due Wednesday, February 26

In this homework, all cohomology groups are over $\mathbb{Z}/2$.

1. We know that, for any space X, its cohomology $H^*(X)$ is a (left) module over the Steenrod algebra $\mathcal{A}$.
(a) Show that, as $\mathcal{A}$-modules, $H^*(X) \cong H^0(X) \oplus H^{>0}(X)$, where the second term is the direct sum of cohomology groups in all positive degrees.
(b) How does $\mathcal{A}$ act on $H^*(\mathbb{S}^n)$? Show that

$$H^*(X \times \mathbb{S}^n) \cong H^*(X) \oplus H^*(X)$$

as $\mathcal{A}$-modules, for any space X.
(c) If $Y \subset X$ is a retract of X, then $\mathcal{A}$-module $H^*(Y)$ is naturally a direct summand of $\mathcal{A}$-module $H^*(X)$.
(d) If spaces X, Y are well-pointed, the usual isomorphism

$$\widetilde{H}^*(X \vee Y) \cong \widetilde{H}^*(X) \oplus \widetilde{H}^*(Y)$$

is an isomorphism of $\mathcal{A}$-modules.

2. Among the following spaces, select those for which the Steenrod algebra $\mathcal{A}$ acts trivially on their cohomology (we say that an action is trivial if Sq^n acts by 0 for all $n > 0$):
(a) A closed oriented surface of genus 3
(b) Suspension of a closed oriented surface of genus 2
(c) 5-dimensional torus
(d) Klein bottle
(e) $\mathbb{S}^2 \times \mathbb{S}^3 \times \mathbb{S}^4$
(f) $\mathbb{S}^2 \vee \Sigma\mathbb{CP}^2$
(g) $K(\mathbb{Z}/3, 1)$
(h) $\mathbb{RP}^5/\mathbb{RP}^3$
Explain your answer.

3. (a) Compute the action of Sq^n for all n on the element $x_1^3 x_2^2 x_3$ in the cohomology ring of $(\mathbb{RP}^\infty)^{\times n}$.
(b) How does Sq^n act on β^i where β is the generator of $H^2(\mathbb{CP}^\infty)$ - can you give a closed formula for the action?

4. The space $\mathbb{CP}^3 \times \mathbb{RP}^3$ has a unique nontrivial cohomology class in degree 3. Compute the action of Sq^n for all n on this class.

5. Determine how the following elements decompose in the basis of admissible monomials:

$$Sq^2 Sq^6, Sq^3 Sq^3, Sq^4 Sq^5, Sq^1 Sq^2 Sq^4, S(Sq^4), S(Sq^5), S(Sq^4 Sq^2),$$

where S is the antipode of the Steenrod algebra.

6. We proved in class that $\mathcal{A}$ acts faithfully on the cohomology of the wedge of $(\mathbb{RP}^\infty)^{\times n}$, over all n. Show that $\mathcal{A}$ does not act faithfully on the cohomology of the wedge of $(\mathbb{RP}^2)^{\times n}$, over all n. (Hint: look for an element of $\mathcal{A}$ that acts by zero on the cohomology of this direct product for any n.)

(7-8). Hatcher, Section 4.L exercises 2 and 5 on pages 517-518.