Name:

Representations of finite groups. Final exam.

In all problems below only complex representations are considered.

1. (30 points) Mark those squares that are followed by correct statements.

☐ Any representation of a finite group is completely reducible.

☐ The alternating group A_4 has five isomorphism classes of irreducible representations.

☐ The tensor product $V \otimes W$ of representations of G is an irreducible representation of $G \times G$ if and only if both V and W are irreducible.

☐ Representations V and W of a finite group G are isomorphic if and only if they have equal characters.

☐ If any irreducible representation of G is one-dimensional then G is cyclic.

☐ The group $SU(2)$ has a subgroup of order 100.

☐ Induced representation $U \uparrow G$ is trivial if U is a trivial representation of $H \subset G$.

☐ Direct sum of two irreducible representations is reducible.

☐ A subrepresentation of a faithful representation is faithful.

2. (20 points) (a) Give the definition of a representation V of a group G.

(b) State the row orthogonality relations for irreducible characters of a finite group G.

3. (10 points) Give an example of a group G and a representation V which is reducible but not completely reducible.

4. (50 points) (a) Write down all irreducible representations of the symmetric group S_4 and the character table.

(b) Which of these representations are faithful?

(c) What are the multiplicities of irreducible representations in the regular representation of S_4?

(d) Compute the inner product $\langle \chi_{reg}, \chi_{reg} \rangle$ where χ_{reg} is the character of the regular representation of S_4.

(e) Pick an irreducible 3-dimensional representation V of S_4 and determine multiplicities of irreducible representations in $V \otimes V$ and $S^2(V)$.

(f) Consider the inclusion of groups $A_4 \subset S_4$. Determine multiplicities of irreducible representations of S_4 in the representations induced from irreducible representations of A_4. Write down the Frobenius matrix.

5. (10 points) What can you say about a representation V given that $\langle \chi_V, \chi_V \rangle = 2$?

6. (20 points) (a) State Frobenius reciprocity law.

(b) State the formula for the dimension of the induced representation $U \uparrow G$, where U is a representation of a subgroup $H \subset G$.

7. (20 points) List all affine graphs together with numbers d_i assigned to their vertices.

8. (20 points) (a) Give an example of a group G and an irreducible representation V such that $\Lambda^2(V)$ is irreducible.

(b) Same problem but with $\Lambda^2(V)$ reducible.

Extra credit: Does there exist a finite group which has exactly ten isomorphism classes of irreducible representations, of dimensions $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, respectively? Justify your answer.