Name:

**Representations of finite groups. Final exam.**

In all problems below only complex representations are considered.

1. (30 points) Mark those squares that are followed by correct statements.
   - □ Any representation of a finite group is completely reducible.
   - □ The alternating group $A_4$ has five isomorphism classes of irreducible representations.
   - □ The tensor product $V \otimes W$ of representations of $G$ is an irreducible representation of $G \times G$ if and only if both $V$ and $W$ are irreducible.
   - □ Representations $V$ and $W$ of a finite group $G$ are isomorphic if and only if they have equal characters.
   - □ If any irreducible representation of $G$ is one-dimensional then $G$ is cyclic.
   - □ The group $SU(2)$ has a subgroup of order 100.
   - □ Induced representation $U \uparrow^G$ is trivial if $U$ is a trivial representation of $H \subset G$.
   - □ Direct sum of two irreducible representations is reducible.
   - □ A subrepresentation of a faithful representation is faithful.

2. (20 points) (a) Give the definition of a representation $V$ of a group $G$.
   (b) State the row orthogonality relations for irreducible characters of a finite group $G$.

3. (10 points) Give an example of a group $G$ and a representation $V$ which is reducible but not completely reducible.

4. (50 points) (a) Write down all irreducible representations of the symmetric group $S_4$ and the character table.
   (b) Which of these representations are faithful?
   (c) What are the multiplicities of irreducible representations in the regular representation of $S_4$?
   (d) Compute the inner product $\langle \chi_{reg}, \chi_{reg} \rangle$ where $\chi_{reg}$ is the character of the regular representation of $S_4$. 
(e) Pick an irreducible 3-dimensional representation $V$ of $S_4$ and determine multiplicities of irreducible representations in $V \otimes V$ and $S^2(V)$.

(f) Consider the inclusion of groups $A_4 \subset S_4$. Determine multiplicities of irreducible representations of $S_4$ in the representations induced from irreducible representations of $A_4$. Write down the Frobenius matrix.

5. (10 points) What can you say about a representation $V$ given that $\langle \chi_V, \chi_V \rangle = 2$?

6. (20 points) (a) State Frobenius reciprocity law.
   (b) State the formula for the dimension of the induced representation $U \uparrow G$, where $U$ is a representation of a subgroup $H \subset G$.

7. (20 points) List all affine graphs together with numbers $d_i$ assigned to their vertices.

8. (20 points) (a) Give an example of a group $G$ and an irreducible representation $V$ such that $\Lambda^2(V)$ is irreducible.
   (b) Same problem but with $\Lambda^2(V)$ reducible.

Extra credit: Does there exist a finite group which has exactly ten isomorphism classes of irreducible representations, of dimensions 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, respectively? Justify your answer.