# McKay Correspondence

Note Title 2009/11/23

## §1. Finite Subgroups of $SU(2)$

### • Conjugacy classes in $U(n)$

Recall the following standard fact from linear algebra:

Any matrix in $U(n)$ is conjugate to a diagonal matrix, i.e. $\forall A \in U(n)$, $\exists B \in U(n)$ s.t. $BAB^{-1} = diag(\lambda_1, \cdots, \lambda_n)$, where $\lambda_i \in S^1 \subseteq \mathbb{C}^*$. The $\lambda_i$'s are nothing but the eigenvalues of $A$.

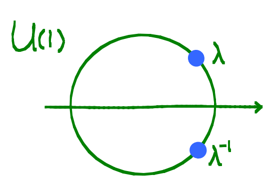
Note that the order of $\lambda_i$'s is unimportant since permuting $\lambda_i$'s can be realized by conjugating by a matrix in $U(n)$.

Consequently, the conjugacy class of a matrix $A \in U(n)$ is determined by the unordered set of its eigenvalues.

The same story applies to $SU(n)$ without much effort: the conjugacy class of a matrix $A \in SU(n)$ is determined by its unordered set of eigenvalues $\{\lambda_1, \cdots, \lambda_n \mid \lambda_1 \cdots \lambda_n = 1\}$. In fact, $U(n)$ is generated by $SU(n)$ and $U(1) = \{\lambda \cdot Id \mid |\lambda| = 1\}$, but under conjugation, $U(1)$ acts trivially.

E.g. $SU(2)$

In this case, the conjugacy classes are parametrized by $\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid |\lambda| = 1\} \cong U(1)$, upto identification of $\lambda$ and $\bar{\lambda} = \lambda^{-1}$.




### • The conjugation representation of $U(n)$ on $Mat(n, \mathbb{C})$.

Let $U(n)$ act on $Mat(n, \mathbb{C})$ by conjugation. We shall decompose it into (real) irreducible subrepresentations.

The first observation to make is that $U(n)$ preserves the subspaces of Hermitian and anti-Hermitian matrices :

$$\mathrm{Mat}(n,\mathbb{C}) \cong (\text{Hermitian}) \oplus (\text{Anti-Hermitian})$$
$$B \longmapsto \left( \frac{B+B^*}{2}, \frac{B-B^*}{2} \right)$$

and the $U(n)$ action preserves these subspaces:

$$B \text{ Hermitian}, A \in U(n) \Longrightarrow (ABA^*)^* = (A^*)^* B^* A^* = ABA^*$$
$$B \text{ anti-Hermitian}, A \in U(n) \Longrightarrow (ABA^*)^* = (A^*)^* B^* A^* = -ABA^*$$

Notation:

$W_+ \triangleq$ the space of Hermitian matrices

$W_- \triangleq$ the space of anti-Hermitian matrices

Note that these are only real representations of $U(n)$ and multiplication by $i$ is an isomorphism of real $U(n)$-modules:

$$W_+ \underset{\cdot i}{\overset{\cdot i}{\rightleftarrows}} W_-$$

We can do a little better, since conjugation preserves traces:

$$W_+ = \{\text{Trace } 0 \text{ Hermitian matrices}\} \oplus \{\lambda \mathrm{Id} \mid \lambda \in \mathbb{R}\}$$
$$B \longmapsto \left( B - \tfrac{\mathrm{tr}B}{n}\mathrm{Id}, \tfrac{\mathrm{tr}B}{n}\mathrm{Id} \right)$$

Denote $W_+^0 \triangleq \{\text{Trace } 0 \text{ Hermitian matrices}\}$, $W_-^0 \triangleq \{\text{Trace } 0 \text{ anti-Hermitian matrices}\}$.

We have shown that:

Lemma 1: As real representations of $U(n)$

$$\mathrm{Mat}(n,\mathbb{C}) \cong \mathbb{R}\cdot\mathrm{Id} \oplus W_+^0 \oplus i\mathbb{R}\mathrm{Id} \oplus W_-^0$$ □

- The double cover $SU(2) \longrightarrow SO(3)$

We shall try to relate finite subgroups of $SU(2)$ to finite subgroups of $SO(3)$, since $SO(3)$, being the rotational symmetry group of the unit sphere $S^2$ in $\mathbb{R}^3$, is more intuitive.

Consider a $2\times 2$ invertible complex matrix

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose inverse is

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Thus for such an $A$ to lie in $SU(2)$, we just need

$$\begin{cases} \det A = 1 \\ \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \end{cases}$$

i.e. $\bar{a}=d$, $c=-\bar{b}$, and $\det A = ad-bc = |a|^2+|b|^2=1$. Write $a=x_1+ix_2$, $b=x_3+ix_4$.
This identifies $SU(2)$ as

$$SU(2)=\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a,b\in\mathbb{C} \right\}$$
$$\cong \{ x_1^2+x_2^2+x_3^2+x_4^2=1 \mid x_i\in\mathbb{R} \} = S^3 \subseteq \mathbb{R}^4,$$

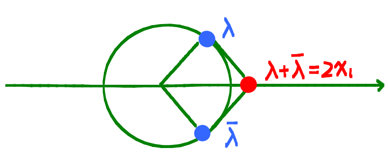
the unit 3-sphere in the 4-dim'l Euclidean space.

Continuting our earlier example of conjugacy classes of $SU(2)$, we see that, the conjugacy class of $A\in SU(2)$ is completely determined by

$$\mathrm{Tr}(A) = a+\bar{a} = 2x_1$$

Conversely, the set of eigenvalues $\{\lambda,\bar{\lambda}\}$ is determined by:

$$\{ x_1 \pm i\sqrt{1-x_1^2} \}$$

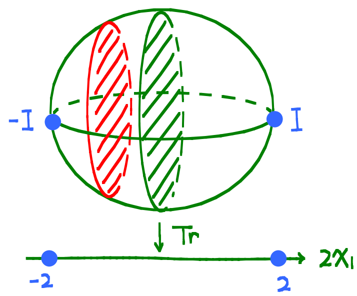



: $\lambda, \bar{\lambda}$ are roots of the equation
$x^2 - \mathrm{Tr}(A)x + \det A = x^2 - 2x_1\cdot x + 1 = 0$

Thus, each conjugacy class of $SU(2)$ is just the set of matrices in $SU(2)$ with a fixed trace value $2x_1$ $(-1\le x_1\le 1)$, which is identified as

$$\{ x_2^2+x_3^2+x_4^2 = 1-x_1^2 \} \cong S^2(\sqrt{1-x_1^2})$$

a two sphere of radius $\sqrt{1-x_1^2}$. (When $x_1 = \pm 1$, this says that the matrices $\pm I$ form their own conjugacy class). Pictorially:




Imagine this to be the 3-sphere of $SU(2)$. The conjugacy classes are just preimages of $tr(A) = 2x_1$. Except for the "poles" $\pm I$, these conjugacy classes are topologically 2-spheres with various radii.

Alternatively, $SU(2)$ can be described as the group of unit quaternions. In fact, quaternions $\mathbb{H}$ may be identified as

$$\mathbb{H} \cong \mathbb{R}\cdot Id \oplus W_-^0 \subseteq Mat(2,\mathbb{C})$$

$$1 \longmapsto Id$$

$$i \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \triangleq \underline{i} \qquad j \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \triangleq \underline{j} \qquad k \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \triangleq \underline{k}.$$

Recall that we have shown: under the conjugation action of $SU(2)$, $W_-^0$ is invariant. Now

$$\begin{aligned} W_-^0 &= \{\text{Traceless anti-hermitian matrices}\} \\ &\cong \mathbb{R}\,\underline{i} \oplus \mathbb{R}\,\underline{j} \oplus \mathbb{R}\,\underline{k} \end{aligned}$$

Moreover

$$\begin{aligned} \{A \in SU(2) \mid TrA = 0\} &= \left\{ \begin{pmatrix} x_1 + i x_2 & x_3 + i x_4 \\ -x_3 + i x_4 & x_1 - i x_2 \end{pmatrix} \,\middle|\, 2x_1 = 0,\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\} \\ &= \left\{ x_2 \underline{k} + x_3 \underline{j} + x_4 \underline{i} \,\middle|\, x_2^2 + x_3^2 + x_4^2 = 1 \right\} \end{aligned}$$

being a conjugacy class in $SU(2)$, is also the unit sphere in $W_-^0$.

$\Longrightarrow$ The conjugation action of $SU(2)$ on $W_-^0$ acts transitively on the "unit sphere" = {traceless elements in $SU(2)$}.

Now we are close to what we need: We have produced a representation $SU(2) \longrightarrow GL(W_-^0) \cong GL(3,\mathbb{R})$ so that $SU(2)$ preserves the "unit sphere" of $\mathbb{R}^3$. We shall impose a norm on $W_-^0$ so that $SU(2)$ preserves the norm, and the "unit sphere" becomes the genuine unit sphere under this norm.

By the same consideration as for $U(n) \cap Mat(n, \mathbb{C})$, a natural candidate for the norm is $(X, Y)' \triangleq Tr(XY)$ since $Tr$ on $W^0_-$ is preserved under conjugation action by $SU(2)$: $\forall A \in SU(2)$, $X, Y \in W^0_-$:

$$\begin{aligned}(A\cdot X, A\cdot Y)' &= Tr(AXA^{-1}\cdot AYA^{-1}) \\ &= Tr(AXYA^{-1}) \\ &= Tr(XY) \\ &= (X, Y)'\end{aligned}$$

and it's readily seen to be bilinear. Furthermore, we have:

$$\begin{cases} Tr(\underline{i}\cdot\underline{i}) = -2 & Tr(\underline{i}\cdot\underline{j}) = 0 \\ Tr(\underline{j}\cdot\underline{j}) = -2 & Tr(\underline{i}\cdot\underline{k}) = 0 \\ Tr(\underline{k}\cdot\underline{k}) = -2 & Tr(\underline{j}\cdot\underline{k}) = 0. \end{cases}$$

Hence if we rescale $(X, Y) \triangleq -\frac{1}{2}(X, Y)' = -\frac{1}{2}Tr(XY)$, we obtain a Euclidean inner product on $W^0_-$, w.r.t. which $\{\underline{i}, \underline{j}, \underline{k}\}$ forms an o.n.b.

Combining the above discussion, we have exhibited a map:

$$SU(2) \longrightarrow Aut(W^0_-, (\ ,\ )) \cong Aut(\mathbb{R}^3, (\ ,\ )) = O(3).$$

It remains to show that:

(i). The image of $SU(2)$ lies in $SO(3)$;
(ii). The map $SU(2) \xrightarrow{\gamma} SO(3)$ is surjective;
(iii). Analyze $Ker\,\gamma = ?$

(i) is easily guanranteed by the topology of $SU(2)$: The group homomorphism $SU(2) \longrightarrow O(3)$ is continuous (i.e. elements in $SU(2)$ close to Id moves vectors in $W^0_-$ only a little), and $SU(2)$ is connected, being a sphere. Thus its image in $O(3)$ must be connected. Since $SO(3)$ is the connected component of $O(3)$ containg $1$, $im(SU(2)) \subseteq SO(3)$. We shall denote the homomorphism:

$$\gamma: SU(2) \longrightarrow SO(3).$$

Exercise: Show that any continuous homomorphism $S^1 \longrightarrow G$, where $G$ is a discrete group, is the trivial one.

(ii). To show that $\gamma$ is surjective, consider the action of

$$A(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \qquad (0 \le \varphi < \pi)$$

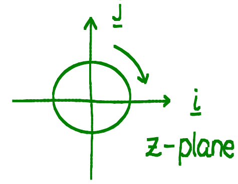
on $W_-^0$:

$$A(\varphi)\cdot \underline{k} = A(\varphi)\underline{k}A(-\varphi) = \underline{k}$$

and on the plane $\mathbb{R}\underline{i} \oplus \mathbb{R}\underline{j} = \left\{ \begin{pmatrix} 0 & i\bar{z} \\ iz & 0 \end{pmatrix} \middle| z \in \mathbb{C} \right\} \cong \mathbb{C}$, $A(\varphi)$ acts as

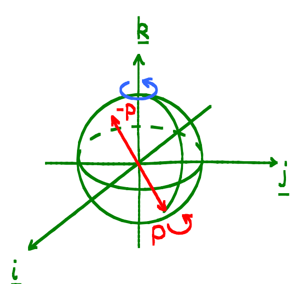
$$A(\varphi)\begin{pmatrix} 0 & i\bar{z} \\ iz & 0 \end{pmatrix} A(-\varphi) = \begin{pmatrix} 0 & i\overline{e^{-2i\varphi}z} \\ ie^{-2i\varphi}z & 0 \end{pmatrix}$$

i.e. it acts as clockwise rotation of the complex plane by the angle $2\varphi$.




In summary, $A(\varphi)$ acts on $W^0_-$ as the rotation about the $\underline{k}$-axis by an angle of $2\varphi$ ($0 \le 2\varphi < 2\pi$ : all the rotations).

Next, we have shown that $SU(2)$ acts transitively on the unit sphere $S^2$ of $W^0_-$. Now $\forall P \in S^2$, $\exists B \in SU(2)$ s.t. $B.\underline{k} = B\underline{k}B^{-1} = P$. The the subgroup $BA(\varphi)B^{-1} \subseteq SU(2)$ $(0 \le \varphi < \pi)$ consists of all rotations about the axis through $\{P, -P\}$:




Now we conclude from the well-known fact that $SO(3)$ consists of all rotations about various axis through the origin that $\gamma$ maps $SU(2)$ surjectively onto $SO(3)$.

(iii). What's the kernel of $\nu: SU(2) \longrightarrow SO(3)$?

$A \in \operatorname{Ker}\nu \Leftrightarrow A$ acts trivially on $W_-^0$ : $\forall X \in W_-^0$, $AXA^{-1} = X$.

(But $A$ commutes with $i\mathbb{R}\mathrm{Id}$ trivially)

$\Leftrightarrow A$ acts trivially on $W_- = i\mathbb{R}\mathrm{Id} \oplus W_-^0$.

$( \cdot i : W_- \xrightarrow{\cong} W_+$ : an isomorphism of $SU(2)$ rep's$)$

$\Leftrightarrow A$ acts trivially on $\mathrm{Mat}(2,\mathbb{C}) \cong W_+ \oplus W_-$.

$\Leftrightarrow A \in Z(\mathrm{Mat}(2,\mathbb{C})) \cap SU(2) = \mathbb{C}\cdot\mathrm{Id} \cap SU(2) = \{\pm I\}$.

Ex. Show that $Z(SU(n)) = \{ \zeta^k \cdot \mathrm{Id} \mid \zeta = e^{\frac{2\pi i}{n}},\ 0 \le k \le n-1 \}$.

In summary, we have shown:

Thm. 2. There exists a 2:1, surjective group homomorphism

$$\nu: SU(2) \longrightarrow SO(3)$$

of (Lie) groups, with $\operatorname{Ker}\nu = \{\pm \mathrm{Id}\}$. □

- Finite subgroups of $SO(3)$.

We shall use the following well-known:

Thm 3. Finite subgroups of $SO(3)$ are classified as follows:

There are two infinite families:

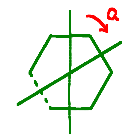
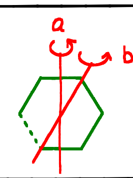
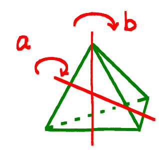
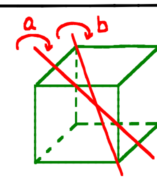
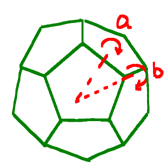
- $C_n$ : cyclic group of order $n$.
- $D_{2n}$ : dihedral group of order $2n$ ;

and 3 more exceptional cases:

- $A_4$ : the rotational symmetry group of a tetrahedron.
- $S_4$ : the rotational symmetry group of a cube / octahedron
- $A_5$ : the rotational symmetry group of an icosahedron / dodecahedron.

For a proof, see M. Artin : Algebra. □

More geometrically, we have the following presentation of these groups:

| $G \subseteq SO(3)$ | $\lvert G \rvert$ | Geometric description of generators |
|---|---|---|
| $C_n = \langle a \mid a^n = 1 \rangle$ | $n$ |   |
| $D_n = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$ | $2n$ |   |
| $A_4 = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$ | $12$ |   |
| $S_4 = \langle a, b \mid a^2 = b^3 = (ab)^4 = 1 \rangle$ | $24$ |   |
| $A_4 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$ | $60$ |   |

## • Finite subgroups of $SU(2)$

Observe that in $SU(2)$, there is only one element of order 2, namely $-I$. This is because any matrix $A \in SU(2)$ can be conjugated to a diagonal matrix of the form $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ and for it to be of order 2, $\lambda = -1$. In contrast, there are lots of elements of order 2 in $SO(3)$ (take any rotation by $\pi$ about any direction in $\mathbb{R}^3$!). Thus the preimages of these order 2 elements in $\mathbb{R}^3$ under $\gamma$ are all of order 4.

Now, let $G$ be a finite subgroup of $SU(2)$ and $H = \gamma(G)$ be its image in $SO(3)$. Since $\nu$ is 2:1, there are two possibilites

(i). $|G|=|H|$, and $\nu|_G : G \xrightarrow{\cong} H$.

(ii). $|G|=2|H|$, and $-I \in G$, $H \cong G/\{\pm I\}$

Also note that from our classification list for $SO(3)$, $|H|$ is even unless $H \cong C_{2k+1}$ is cyclic of odd order. Other than this $|H|$ is even $\Rightarrow |G|$ is even $\Rightarrow G$ has an order 2 element (elementary group theory!), and we are in case (ii).

We analyse case by case

(a). $H \cong C_n$. There are two possibilites:

(a.i) $n=2k+1$. Then $G \cong H \cong C_{2k+1}$ or $G \cong C_2 \times H \cong C_{2(2k+1)}$

(a.ii) $n=2k$. Then $G/\{\pm I\} \cong H \cong C_{2k} \Rightarrow G \cong C_{4k}$ or $C_2 \times C_{2k}$. The latter is ruled out since there would be more than 1 order 2 elements in $G$.

Thus $G$ is always cyclic. Such $G$ is always conjugate to one of the form:

$$G = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \middle| \, \zeta = e^{\frac{2\pi i k}{n}},\ 0 \le k < n \right\}$$

(b). $H \cong D_{2n} \Rightarrow H \cong G/\{\pm I\}$. In this case $H \supseteq C_n$ as a (normal) subgroup $\Rightarrow \nu^{-1}(C_n) \cong C_{2n} \subseteq G$ (by case (a)), of index 2, and thus must be normal. Since $D_{2n} = C_n \amalg a \cdot C_n$ ($a$ of order 2) $\Rightarrow G = \nu^{-1}(H) = C_{2n} \amalg a' C_{2n}$, where $\nu(a') = a$ and $a'$ must have order 4. Then $G$ can be conjugated to the group generated by:

$$\left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = a' \,\middle|\, \zeta = e^{\frac{\pi k i}{n}},\ 0 \le k < 2n \right\}$$

We denote this group by $D_{2n}^*$, called the binary dihedral group. Note that $D_{2n}^* \ncong D_{4n}$ since the latter has many order 2 elements.

$$\begin{array}{ccc} D_{2n}^* & \xrightarrow{\nu} & D_{2n} \\ \triangledown\!/ & & \triangledown\!/ \\ C_{2n} & \xrightarrow{\nu} & C_n \end{array} \qquad \text{with } \nu\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = 1$$

Rmk: It's not hard to figure out the structure of $D_{2n}^*$ directly from elementary group theory: Let $t$ be the generator of $C_{2n}$. $s = \nu(t) \in C_n$ ($\subseteq D_{2n}$) a generator. Then $\nu(a')\nu(t)\nu(a'^{-1}) = asa^{-1} = s^{-1} = \nu(t^{-1}) \Rightarrow a'ta'^{-1} = t^{-1}$ or $-t^{-1}$. But if $a'ta'^{-1} = -t^{-1}$ $\Rightarrow (a't)^2 = -t^{-1}a'a't = 1 \Rightarrow a't = \pm 1 = \mp(a')^2 \Rightarrow t = \mp a' \Rightarrow G$ is abelian. Contradiction. So $a'ta'^{-1} = t^{-1}$ and it's isomorphic to the group above.

(c). $H \cong A_4, S_4, A_5. \Rightarrow H \cong G/\{\pm I\}$. In these cases the corresponding $G$'s are denoted $A_4^*, S_4^*, A_5^*$, called the binary tetrahedron group, binary octahedron group, binary icosahedron group respectively.

Rmk: Note that $A_4^* \ncong S_4$, $A_5^* \ncong S_5$, since $S_4, S_5$ have more than 1 order 2 elements.

By now, we have classified all finite subgroups of $SU(2)$:

Thm 4. Finite subgroups of $SU(2)$ are classified as follows:

| $G \subseteq SU(2)$ | Presentation | $\lvert G \rvert$ |
|---|---|---|
| $C_n$ | $\langle a \mid a^n = 1 \rangle$ | $n$ |
| $D_{2n}^*$ | $\langle a, b \mid a^2 = b^2 = (ab)^n$ | $4n$ |
| $A_4^*$ | $\langle a, b \mid a^2 = b^3 = (ab)^3$ | 24 |
| $S_4^*$ | $\langle a, b \mid a^2 = b^3 = (ab)^4$ | 48 |
| $A_5^*$ | $\langle a, b \mid a^2 = b^3 = (ab)^5$ | 120 |

$\square$

## §2. The McKay Graph

Let $V \cong \mathbb{C}^2$ be the 2-dimensional representation of $SU(2)$. By restricting it to any finite subgroup $G$ of $SU(2)$, we obtain a 2-dim'l representation of $G$, still denoted by $V$. Note that $V$ is irreducible unless $G \cong C_n$, the only finite abelian subgroups of $SU(2)$ (otherwise $V \cong U \oplus W$ is a sum of 2 1-dim'l representations $\Longrightarrow G \subseteq \mathbb{C}^* \times \mathbb{C}^*$ is abelian). This representation plays a pivotal role in what follows.

Lemma 5. $V$ is a self-dual representation.

Pf: $\forall g \in G$, $\exists B \in SU(2)$ s.t. $BgB^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $|\lambda| = 1$. Thus

$$\chi_V(g) = \mathrm{tr}_V(g) = \mathrm{tr}_V(BgB^{-1}) = \lambda + \lambda^{-1} = \lambda + \bar{\lambda} \in \mathbb{R}$$

$\Longrightarrow \chi_V$ is real $\Longrightarrow V$ is self-dual. $\square$

Rmk: Using character theory for connected compact Lie groups, we can see $V$ is a self-dual representation for $SU(2)$. Such an isomorphism $V \longrightarrow V^*$ is not hard to exhibit:

$\forall\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2) \Longrightarrow g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Longrightarrow (g^{-1})^t = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$. Let $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then

$$h^{-1} g h = (g^{-1})^t = g^*.$$

Let $V_i, V_j$ be two irrep's of $G$. Consider the multiplicity of $V_i$ in $V_j \otimes V$.

$$m(V_i, V \otimes V_j) = \dim \mathrm{Hom}_G(V_i, V \otimes V_j) = \dim \mathrm{Hom}_G(V \otimes V_j, V_i).$$

Lemma 6. $m(V_i, V \otimes V_j) = m(V_j, V \otimes V_i)$.

Pf: Since $m(V_i, V \otimes V_j) \in \mathbb{Z}_{\geq 0}$, $m(V_i, V \otimes V_j) = \overline{m(V_i, V \otimes V_j)}$. $\Longrightarrow$

$$\begin{aligned} \frac{1}{|G|} \textstyle\sum_{g \in G} \chi_i(g) \overline{\chi_V(g) \chi_j(g)} &= (\chi_i, \chi_V \chi_j) \\ &= m(V_i, V \otimes V_j) \\ &= \overline{m(V_i, V \otimes V_j)} \\ &= \frac{1}{|G|} \textstyle\sum_{g \in G} \overline{\chi_i(g)\, \overline{\chi_V(g) \chi_j(g)}} \\ &= \frac{1}{|G|} \textstyle\sum_{g \in G} \overline{\chi_i(g)}\, \chi_V(g) \chi_j(g) \end{aligned}$$

$$= \frac{1}{|G|} \sum_{g\in G} \overline{\chi_i(g)\chi_V(g)}\, \chi_j(g) \qquad (V \text{ is self-dual})$$
$$= (\chi_j, \chi_i \chi_V)$$
$$= m(V_j, V\otimes V_i). \qquad \square$$

Rmk: In general, it's true that $\forall X, Y, Z$ rep's of $G$,

$$\mathrm{Hom}_G(X, Z\otimes Y) \cong \mathrm{Hom}_G(X\otimes Z^*, Y)$$

If $X \cong V_i$, $Y\cong V_j$, $Z\cong V\cong V^*$, taking dimension of both sides, we obtain:

$$\begin{aligned} m(V_i, V\otimes V_j) &= \dim \mathrm{Hom}_G(V_i, V\otimes V_j) \\ &= \dim \mathrm{Hom}_G(V_i\otimes V, V_j) \\ &= m(V_j, V_i\otimes V). \end{aligned}$$

- Construction of the graph

Notation: $a_{ij} \triangleq m(V_i, V\otimes V_j)$. Then $a_{ij} = a_{ji}$.

Now to each finite subgroup $G$ of $SU(2)$, we associate with it a graph $\Gamma$ as follows:

Vertices: Irrep's $V_i$ of $G$.

Edges: The $i, j$ th vertices are connected by $a_{ij}$ edges.

Moreover, to each vertex, we assign to it a weight $d_i = \dim V_i$.

E.g. $G \cong C_n = \langle a \mid a^n = 1\rangle$.

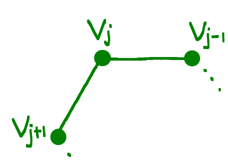
We know that in this case, Irrep's of $G$ are all 1 dimensional:

$$\mathrm{Irrep}(G) = \{V_0, V_1, \cdots, V_{n-1}\}$$

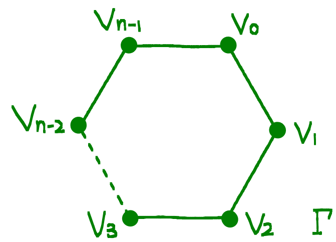
where $a$ acts on $V_k$ by multiplication by $\zeta^k = e^{\frac{2\pi k i}{n}}$, $0\le k<n$. Moreover, since $a = \begin{pmatrix} \zeta & \\ & \zeta^{-1}\end{pmatrix}$, we see that $V \cong V_1 \oplus V_{-1}$ ($V_i = V_{n+i}$). Thus $\forall V_j$

$$V_j \otimes V \cong V_j \otimes (V_1 \oplus V_{-1}) \cong V_{j+1} \oplus V_{j-1}.$$

Hence in the graph:



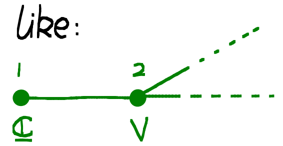
and the graph looks like:



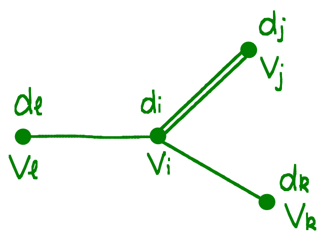

• Common features of McKay graphs.

Now we discuss about general properties of the graph.

Note that for $G$ non-abelian, $V$ is irreducible. $\mathbb{C} \otimes V \cong V$. Thus $\Gamma$ always contains a portion like:




For any vertex $V_i$, consider all the vertices connected to it:




Then by definition, $V_i \otimes V \cong \oplus V_j^{a_{ij}}$. Taking dimension of both sides, we get:

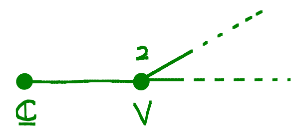
$$2d_i = \Sigma\, a_{ij} d_j.$$

Later we will show that, except for two degenerate cases, vertices in any McKay graph are connected by at most 1 edge.

Thm. 7. McKay graphs are connected.

Pf: By the example above, it suffices to prove for $G$ nonabelian. We shall prove by contradiction.

Assume for some $G$, $\Gamma$ is not connected. Then, by our discussion above, $\exists$ irrep $V_i$ of $G$ not contained in the connected component of




Note that the irreps of $G$ occuring in this component are precisely those irreps occuring inside $V^{\otimes n}$ for various $n \in \mathbb{Z}_{\geq 0}$ (by definition). Thus such $V_i$ must satisfy:

$$
\begin{aligned}
& (\chi_i, \chi_{V^{\otimes n}}) = 0, \quad \forall n \geq 0 \\
\Longleftrightarrow\ & (\chi_i, \chi_V^n) = 0, \quad \forall n \geq 0 \\
\Longleftrightarrow\ & \tfrac{1}{|G|} \textstyle\sum_g \chi_i(g) \overline{\chi_V(g)}^n = 0 \\
\Longleftrightarrow\ & \tfrac{1}{|G|} \textstyle\sum_g \chi_i(g) \chi_V(g)^n = 0. \quad (V \text{ is self-dual})
\end{aligned}
$$

By earlier discussion in §1, $\chi_V(g) \in [-2, 2]$ and $\chi_V(g) = -2$ iff $g = -I$, $\chi_V(g) = 2$ iff $g = I$. Since we have assumed that $G$ is non-abelian, $-I \in G$. Divide both sides of the equation by $2^n$, and multiplying by $|G|$, we obtain:

$$
\begin{aligned}
& \textstyle\sum_g \chi_i(g) \left(\frac{\chi_V(g)}{2}\right)^n = 0, \quad \forall n \geq 0 \\
\Longleftrightarrow\ & \chi_i(I) + \chi_i(-I)(-1)^n + \textstyle\sum_{|\chi_V(g)/2| < 1} \chi_i(g) \left(\frac{\chi_V(g)}{2}\right)^n = 0, \quad \forall n \geq 0
\end{aligned}
$$

Since $-I \in Z(G)$, by Shur's lemma, $-I$ acts on $V_i$ by a scalar matrix. Since $(-I)^2 = I$, it can only act as $\pm \mathrm{Id}_{V_i}$. Hence $\chi_{V_i}(-I) = \mathrm{tr}_{V_i}(\pm \mathrm{Id}_{V_i}) = \pm d_i$. Now, divide both sides of the equation by $d_i$, we have:

$$
1 + \varepsilon (-1)^n + \textstyle\sum_{|\chi_V(g)/2| < 1} \frac{\chi_i(g)}{d_i} \cdot \left(\frac{\chi_V(g)}{2}\right)^n = 0, \quad \forall n \geq 0,
$$

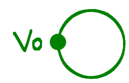
where $\varepsilon = \frac{\chi_i(-I)}{d_i} = \pm 1$ is fixed for $V_i$. Taking $n \gg 0$, since $\left|\frac{\chi_V(g)}{2}\right| < 1$, the rest of the summation is arbitrarily small and has to be an integer, it must be 0. Hence we get an equation for all $n \gg 0$:

$$
1 + \varepsilon (-1)^n = 0
$$

This is impossible and leads to the desired contradiction. $\square$

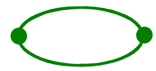
Cor. 8. $a_{ij} \leq 1$ unless $G \cong \{1\}$ or $C_2$.

Pf: $G \cong \{1\} \Longrightarrow \Gamma$ has only 1 vertex, namely $V_0 = \mathbb{C}$. $V \cong V_0 \oplus V_0 \Longrightarrow a_{00} = 2$. $\Gamma$ looks like

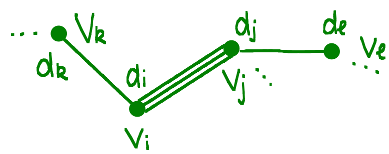



(the edge, considered leaving and entering, connects $V_0$ twice)

For $C_2$, we have shown that its graph is like:



Conversely, assume that $G \ncong \{1\}$, and there is a multiple edge between $V_i$ and $V_j$:




we have $a_{ij} = a_{ji} \geq 2$

$$\begin{cases} 2d_i = a_{ij}d_j + \sum a_{ik}d_k \\ 2d_j = a_{ji}d_i + \sum a_{jl}d_l \end{cases}$$

$\Longrightarrow 2d_i = 2d_j + (a_{ij}-2)d_j + \sum a_{ik}d_k = a_{ji}d_i + (a_{ij}-2)d_j + \sum a_{ik}d_k$

$\Longrightarrow 2(a_{ij}-2)d_j + \sum a_{ik}d_k = 0$

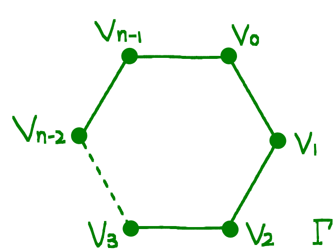
$\Longrightarrow d_k = 0$, $a_{ik} = 0$, $a_{ij} = 2$. i.e. no vertex other than $V_j$ connects to $V_i$.

By symmetry, this must also be true for $V_j$. Since we know that $\Gamma$ is connected, $\Gamma$ must then be:

and $G \cong \mathbb{Z}/2$ (the only group with only 2 conjugacy classes). $\square$

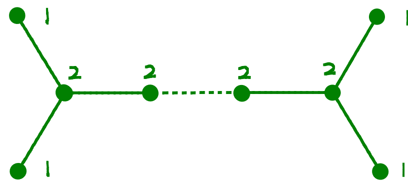
- List of McKay graphs

We have seen that the McKay graph for $C_n$ is




This graph is called $\widetilde{A}_{n-1}$

The graph for $D_{2n}^*$ is the following with $n+3$ vertices:

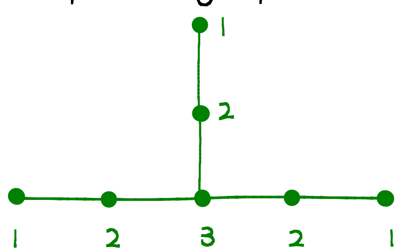



The graph is called $\widetilde{D_{n+2}}$. One can check the relation:

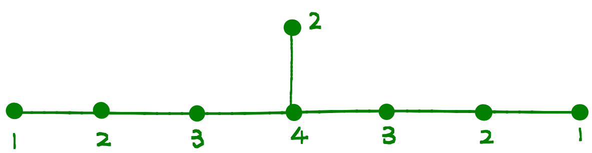
$$|G| = \Sigma d_i^2$$

from: $4n = 4 \cdot 1^2 + (n-1) \cdot 2^2$.

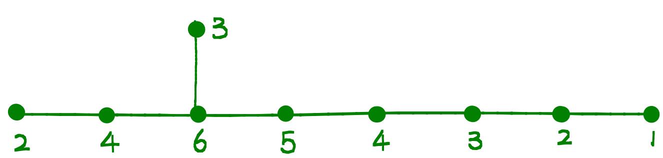
The exceptional groups:




$A_4^*$ : the graph $\widetilde{E}_6$




$S_4^*$ : the graph $\widetilde{E}_7$




$A_5^*$ : the graph $\widetilde{E}_8$

We shall prove, in the next section, that these are the only possibilities:

Thm. Any connected graph $\Gamma$ with positive integral weights $d_i$ assigned to each vertex satisfying:

(i). $g.c.d(d_i) = 1$

(ii). $2d_i = \Sigma_{i-j}\, d_j$

is one of the graphs listed above.