§8. $\mathcal{U}_3$ Link Homology

In this section we define a new link homology theory by categorifying the $U_q(\mathcal{U}_3)$-link invariant.

Recall that the $U_q(\mathcal{U}_3)$-link invariant is the assignment

$$P: \text{Oriented links } \rightarrow \mathbb{Z}[q,q^{-1}]$$

$$L \mapsto P(L)$$

determined by:

1. The Skein relation

$$q^3 P(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}) - q^{-3} P(\begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array}) = (q - q^{-1}) P(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array})$$

2. The normalization condition:

$$P(\begin{array}{c} \circlearrowleft \end{array}) = q^2 + 1 + q^{-2} = [3]$$

$$P(\emptyset) = 1.$$  

The Skein relation implies that, for any oriented link $L$, we have:

$$P(\begin{array}{c} L \cup \bigcirc \end{array}) = [3]. P(L)$$

In particular,

$$P(\underbrace{\bigcirc \bigcirc \cdots \bigcirc}_k) = [3]^k$$

Kuperberg's $\mathcal{U}_3$ spiders.

G. Kuperberg developed a planar graphical calculus for $U_q(\mathcal{U}_3)$
representation theory. (We will talk more about representation theory of quantum groups later. For the moment, one can just regard the quantum group $U_q(\mathfrak{sl}_3)$ as $U(\mathfrak{sl}_3)$, the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_3$.)

The planar graphs of Kuperberg, called webs, are planar oriented trivalent graphs whose vertices look like either

or

For instance, the following are webs:

Note that webs are naturally bipartite. Kuperberg uses these webs to define the $U_q(\mathfrak{sl}_3)$ link invariant, by resolving crossings as:

$$P\left(\begin{array}{c}
\end{array}\right) = q^{-2} P\left(\begin{array}{c}
\end{array}\right) - q^{-3} P\left(\begin{array}{c}
\end{array}\right)$$

$$P\left(\begin{array}{c}
\end{array}\right) = q^{2} P\left(\begin{array}{c}
\end{array}\right) - q^{3} P\left(\begin{array}{c}
\end{array}\right).$$

Webs are subject to the simplifications:

$$(1). \quad \begin{array}{c}
\end{array} = [3]$$
(2). \[ \uparrow \] = \[ 2 \]

(3). \[ \begin{array}{c}
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For this, see Kuperberg: Spiders for Rank 2 Lie Algebras.

From a representation theory point of view, trivalent vertices can be regarded as intertwiners of $U_q(sl_3)$-modules. Namely, to points with $+/-$ signs, we assign the standard $U_q(sl_3)$-module $V$ and its dual $V^*$:

\[ \begin{array}{c}
\bullet^+ & \mapsto & V \\
\bullet^- & \mapsto & V^*
\end{array} \]

A trivalent vertex, after bending a little bit, is assigned to the intertwiner of $U_q(sl_3)$-modules:

\[ V^\otimes 3 \cong \frac{\Lambda^3 V \oplus S^3 V \oplus \ldots}{C} \]

\[ V^{* \otimes 3} \cong \frac{\Lambda^3 V^* \oplus S^3 V^* \oplus \ldots}{C} \]

One readily sees that this assignment is invariant under rotations of the trivalent vertices.

The following thm. of Kuperberg states that, the braided monoidal category of $U_q(sl_3)$-modules is equivalent to the category of webs with boundaries modulo the above simplifications.
Thm. 1. (Kuperberg). (1) We have a braided monoidal category $G$ with

Objects: sequences of signed points:

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+ - ... +
```

Morphisms: $\mathcal{C}(q)$-linear combinations of webs with boundaries:

modulo the simplification relations. The braiding is given by resolutions of the usual braidings:

(2). Moreover, for any sequence of signs $\epsilon, \epsilon'$:

$$\text{Hom}_G(\epsilon, \epsilon') = \text{Hom}_{u_q(sl_3)}(V^{\otimes \epsilon}, V^{\otimes \epsilon'})$$

The $\mathcal{C}(q)$-space $\text{Hom}_G(\epsilon, \epsilon')$ has as basis the elliptic webs, which are webs like below (more complex than hexagons):

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For the proof of thm 1, see the paper of Kuperberg.

E.g. The $\mathbb{C}(q)$-vector space $\text{Hom}(++,++)$ is spanned by

\begin{align*}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\end{array}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\end{array}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\end{array}
\end{align*}

Now, any oriented tangle $T$, after resolution, becomes a linear combination of webs with boundaries, and can thus be regarded as an element in $\text{Hom}_\mathbb{C}(\epsilon,\epsilon')$ for some $\epsilon,\epsilon' \in \text{Ob}(G)$. Thm 1 then tells us that this is a tangle invariant.

Rmk: Kuperberg in fact defined spiders for all rank 2 Lie algebras: $\mathfrak{sl}_3$, $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$, $\mathfrak{g}_2$. The latter ones have similar diagrammatics as $\mathfrak{sl}_3$. But there are rational coefficients for web diagrams that occur in the intermediate steps that make them hard to categorify.

Cobordisms of webs

As for the Jones polynomial, we want to categorify the $\mathfrak{sl}_3$-link invariant $P(L)$ to a functorial (up to $\pm 1$) bigraded homology theory $H(L) = \bigoplus_{i,j} H^{i,j}(L)$, so that

$$
\chi(H(L)) = \sum_{i \in \mathbb{Z}} (-1)^i \text{rank } H^{i,j}(L) \varphi^j = P(L).
$$

To achieve this, we again must assign a commutative Frobenius algebra $A$ to a circle, whose graded rank is

$$
\text{gr. rk } A = [3].
$$
By analogy, we will just take \( A \cong H^*(\mathbb{C}P^2) \cong \mathbb{Z}[x]/(x^3) \), and degrees need to be shifted down accordingly:

\[
A \cong H^*(\mathbb{C}P^2) \{-2\} \cong \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot x^2
\]

Then this determines a 2d TQFT \( F \), which when applied to an oriented surface \( S \) with boundary gives a deg. \(-2\times(S)\) map:

\[
\begin{array}{c}
A^\otimes \ell \\
\downarrow \\
F(S) \\
\downarrow \\
A^\otimes k
\end{array} \quad \rightarrow \quad \begin{array}{c}
\vdots \\
\vdots
\end{array}
\]

However, now we have more than just circles after resolving oriented link diagrams; we have webs \( \Gamma' \). But the fact that \( P(\Gamma') \in \mathbb{Z}[a,a^{-1}] \) gives us a hint that \( H(\Gamma') \), just as for circles, should only live in homological degree 0:

\[
H(\Gamma') \cong \bigoplus_{j \in \mathbb{Z}} H^{0,j}(\Gamma')
\]

Furthermore, we want maps

\[
\begin{array}{c}
\downarrow \\
H
\end{array} \quad \leftrightarrow \quad \begin{array}{c}
\downarrow \\
H
\end{array}
\]

in both directions to take care of the resolution of crossings. In fact, we want chain complexes:

\[
0 \longrightarrow H(\quad \{-2\} \longrightarrow H(\quad \{-3\} \longrightarrow 0,
\]

Homological deg 0
We also need a multiplicative categorification to take care of:
\[ P(\Gamma_1 \cup \Gamma_2) = P(\Gamma_1) \cdot P(\Gamma_2) \]

The most straightforward way to categorify this is to require:
\[ H(\Gamma_1 \cup \Gamma_2) = H(\Gamma_1) \otimes_{\mathbb{Z}} H(\Gamma_2) \]

By the expected functoriality, we need to look for cobordisms between the webs with boundaries:

Such cobordisms are called foams, which are 2-dimensional CW complexes with singularities along circles/arcs.

Near the singularities, foams look like “Y×I”:
Finally, we require each 2-dim'l smooth cell of foams to be oriented so that the induced orientations on the boundaries of the cells agree with the prescribed orientations on the webs.

Together with the usual cobordisms in $\text{Cob}_2$ (birth of a circle, death of a circle, saddle moves), the foams:

generate the cobordisms of webs (with boundaries).

**Cohomology of flag varieties**
Before moving on to the TQFT of foams, we will first look at
a special example of a web. A trick later will enable us to reduce closed foams to this case.

We now try to determine what graded module \( H(\Theta) \) we should assign to the "\( \Theta \)-web":

![The '\( \Theta \)-web']

Note that the three edges of \( \Theta \) allow us to merge circles into it in three ways:

![Three ways to merge circles into \( \Theta \)-web]

Thus we expect \( H(\Theta) \) to be an \( A \)-module in 3 different ways. In general, if a web \( \Gamma \) has \( k \) edges, we would expect \( H(\Gamma) \) to be an \( A^{\otimes k} \)-module.

Next, notice that, by the rule of removing a digon, we have:

\[
P(\Theta) = [3] \cdot [2]
\]

Since \( \text{gr.rk } H^*(\mathbb{C}\mathbb{IP}^1) \equiv [2] \), \( \text{gr.rk } H^*(\mathbb{C}\mathbb{IP}^3) \equiv [3] \), the most natural way to lift \( P(\Theta) \) is to realize it as the cohomology ring of a manifold, which can be regarded as a \( \mathbb{IP}^1 \)-bundle over \( \mathbb{IP}^2 \) in 3 different ways. There is a unique such example, namely the full flag variety \( \text{Fl}_3 \) of the Lie algebra \( \mathfrak{sl}_3 \).

Recall that \( \text{Fl}_3 \) consists of full flags in \( \mathbb{C}^3 \):
\[ \text{Fl}_3 = \{ (V_i)_{i=1}^3 \mid \dim V_i = i, \ 0 \leq V_1 \subseteq V_2 \subseteq \mathbb{C}^3 \} \]

Note that we have $\text{IP}(\mathbb{C}^3) \cong \text{CP}^2$ choices for $V_1$, and for each fixed $V_1$, there is $\text{IP}(\mathbb{C}^3/V_1) \cong \text{CP}^1$ choices for $V_2$. This gives us a description of $\text{Fl}_3$ as a $\text{IP}^1$-bundle over $\text{IP}^2$.

The 3 different $\text{IP}^1$-bundle realizations of $\text{Fl}_3$ are given by regarding $\text{Fl}_3$ as a real analytic manifold. Namely, if we fix a hermitian inner product on $\mathbb{C}^3$, then any full flag $(V_1, V_2)$ is completely determined by first choosing $L_1 = V_1$, and taking the orthogonal complement $L_2$ of $V_1$ in $V_2$, and finally letting $L_3$ be the direction perpendicular to $V_2$ in $\mathbb{C}^3$. In this way, we have set up a bijection of full flags $\text{Fl}_3$ with the collection:

\[ \{ (L_i)_{i=1}^3 \mid L_i \perp L_j, \forall i, j \} \]

We can permute $L_i$'s and start by choosing any of them first ($\text{CP}^2$ worth of choices). Thus we obtain the three different $\text{IP}^1$-bundle descriptions of $\text{Fl}_3$.

In general, if $G$ is a compact Lie group and $T$ is a maximal torus of $G$, the flag variety $\text{Fl}$ of $G$ can be regarded as the homogeneous manifold $G/T$, or holomorphically as $G^\mathbb{C}/B$, where $G^\mathbb{C}$ is the complexification of $G$, and $B$ is a Borel subgroup of $G^\mathbb{C}$. The permutation of $L_i$'s above is realized by the Weyl group $W = \text{N}_G(T)/T$ action on $\text{Fl}$, which doesn't preserve the holomorphic structure of $\text{Fl}$. 
Algebraically,
\[ H^*(F_{13}) \cong \mathbb{Z}[x_1, x_2, x_3] / (x_1 + x_2 + x_3, x_1 x_2 + x_1 x_3 + x_2 x_3, x_1 x_2 x_3) \]
where \( x_i \) is the first Chern class of the line bundle on \( F_{13} \), which assigns to the point \( (L_1, L_2, L_3) \) the line \( L_i \). This is a Frobenius algebra of dimension 6, with basis \( \{ x^r x^s \mid r \leq 2, s \leq 1 \} \). The trace map is:
\[
\varepsilon(x^r x^s) = \begin{cases} 
1 & \text{if } r=2, s=1 \\
0 & \text{otherwise} 
\end{cases}
\]
We take \( H(\Theta) \cong H^*(F_{13}, \mathbb{Z}) \). The Frobenius algebra structure gives us a unit map \( \iota \) (resp. trace map \( \varepsilon \)), which must come from a foam from \( \Phi \) to \( \Theta \) (resp. \( \Theta \) to \( \Phi \)).

Recall that in §2, we introduced the handy graphical notation of depicting "multiplication by elements of \( A \)" by sewing a patch labeled by that element. Now we have patches from \( A = \mathbb{Z}[x_2]/(x_3) \):
and

Thus the unit map with patches:

\[ \ldots \]

\( n \geq 3 \)

denotes the inclusion of the element \( \lambda^r \lambda^s \lambda^t \) into \( H(\Theta) \). Closing it up with the cap:

represents the integer \( \varepsilon(\lambda^r \lambda^s \lambda^t) \). In particular, we have:

\[ \ldots \]

\[ = 1 \]
Also note that by modding out symmetric functions, the inclusion of elements is an anti-symmetrizer:

\[ = 0 \quad (r \neq 2 \text{ or } S \neq 1) \]

\( = - \)

\( = - \)

\( = 0 \quad \text{in } H(\Theta) \)

In particular, any symmetric distribution of dots around a singular circle is always 0:

\( = 0 \)

**Evaluating closed foams**

Now, we recall a trick from §2 that will always allow us to reduce closed foams to closed surfaces and \(\Theta\) bubbles, possibly with dots on them. This is done by locally decomposing any cylinder into patches:

\[ = \sum_{i=1}^{\text{dim}A} \epsilon_{b_i} \]

where \(\{a_i\}, \{b_i\}\) are dual bases under \(\epsilon\). Here for later convenience.
we will take $-E$ as our trace pairing for $A$ (or $A = H^*(CIP^2, \mathbb{Z})$). so that:

$$
\begin{array}{cccc}
\text{ cylinder} & \text{ circle} & \text{ circle} & \text{ circle} \\
\end{array}
$$

Then, inside any (closed) foam, near a singular circle, we can do a “surgery” around a smooth loop winding around the singular circle, and sewing back the above patches:

![Diagram showing surgery process](image)

We will perform a surgery around the purple circle.

Cut out the purple band, which is diffeomorphic to $S^1 \times I$.

(1)

(2). Paste back the patches:

![Paste back patches](image)
(3). Repeat this process to the left wall as well. We will get linear combinations of pictures where a "$\Theta$-bubble" lies in between two walls, all carrying some dots:

\[ \sum (\pm 1) \]

In this way we will always reduce closed foams to the case of closed surfaces and $\Theta$-bubbles carrying dots, to both of which we already know how to assign values. Although it may be impractical, it will help to determine $H(I')$ for webs $I'$, as described below.

**Graded modules for webs**

Being able to evaluate closed foams that carry dots will also enable us to determine the graded module $H(I')$ for any web $I'$, at least in theory.

Indeed, we have a set of generators for $H(I')$, namely, we can regard any element of $H(I')$ as a cobordism from $\Phi$ to
\[ \Gamma', \text{ possibly carrying dots. Now we reverse this process by} \]
\[ \text{taking the free graded abelian group } \tilde{H}(\Gamma') \text{ generated by all } \]
\[ \text{such cobordisms carrying dots.} \]

Then we need to mod out all the relations among generators. A linear combination of the above cobordism pictures will give us a relation iff, whenever we cap off these pictures with any fixed cobordism from \( \Gamma' \) to \( \emptyset \), the combination of closed foams evaluates to 0, using the algorithm we introduced in the previous subsection. We denote all relations so obtained by \( R(\Gamma') \), which is a graded submodule of \( \tilde{A}(\Gamma') \).

\[
\begin{array}{ccc}
\includegraphics[width=0.2\textwidth]{example1.png} & + & \includegraphics[width=0.2\textwidth]{example2.png} \\
\includegraphics[width=0.2\textwidth]{example3.png}
\end{array}
\in R(\Gamma')
\]

By construction, we have:

\[ H(\Gamma) \cong \tilde{A}(\Gamma') / R(\Gamma') . \]

\textbf{E.g.} Some easy \( H(\Gamma') \)'s.

(1). \( \Gamma' = \emptyset \).
Any closed foam $S$ with dots gives a cobordism from $\phi$ to $\phi$, and thus represents an element of $\tilde{A}(\Gamma')$. So

$$\tilde{A}(\phi) \cong \oplus \mathbb{Z}\langle S \rangle$$

But closing up $S$ by any cobordism $S'$ from $\phi (= \Gamma')$ to $\phi$ is just the disjoint union of $S$ with $S'$.

$$S'$$

$$\Gamma = \phi$$

$$S$$

Since $\varepsilon(S \cup S') = \varepsilon(S) \cdot \varepsilon(S')$, it follows that $S - \varepsilon(S) \overline{\phi}$, is in $R(\Gamma')$, where $\overline{\phi}$ denotes the empty cobordism between $\phi$ and $\phi$. We conclude that

$$H(\phi) \cong \mathbb{Z}\langle \overline{\phi} \rangle$$

(2). $\Gamma' = \bigcirc$.

Again, generators of $H(\phi)$ are given by foams from $\phi$ to $\bigcirc$:

Using the surgery trick above, we can always reduce such foams to the case of a cup with no more than 2 dots on it:
The closed foams below the cups are then evaluated using $\varepsilon$. It follows that $H(\mathcal{O})$ is generated by the cups:

\[
\begin{array}{llll}
\includegraphics[width=0.15\textwidth]{example1.png} & = & \includegraphics[width=0.15\textwidth]{example2.png} & - \includegraphics[width=0.15\textwidth]{example3.png} - \includegraphics[width=0.15\textwidth]{example4.png}
\end{array}
\]

Moreover, there is no linear relation among these cups since they have different degrees. We thus recover our earlier assignment $H(\mathcal{O}) = H^*(\mathbb{C}P^2)$, as graded abelian groups.

\[
H(\mathcal{O}) \cong \mathbb{Z}\langle \includegraphics[width=0.15\textwidth]{example5.png} \rangle \oplus \mathbb{Z}\langle \includegraphics[width=0.15\textwidth]{example6.png} \rangle \oplus \mathbb{Z}\langle \includegraphics[width=0.15\textwidth]{example7.png} \rangle
\]

In summary, so far we have defined the graded abelian groups $H(\Gamma)$ for any web $\Gamma$. In the following we will check that this assignment categorifies Kuperberg's spider calculus and define the $\mu$ link homology.

**Categorification of Kuperberg's spiders**

Now we check that our assignment $\Gamma \mapsto H(\Gamma)$ lifts Kuperberg's graphical calculus. Details are left as exercises.

**Lemma 2.** We have the following cobordism decomposition:
Proof omitted. This is done by “capping” and “capping” off both sides of the equation, and doing surgeries on both sides along singular circles. See M. Khovanov, $\mathcal{U}(3)$ Link Homology.

Now we can set up maps between a digon and a line:

Using lemma 2, one checks that this gives an orthogonal decomposition of the identity morphism of the digon. Taking into account of the degree shifts ($\chi(3S) - 2\chi(S) + 2 \cdot \# \text{ dots}$), we obtain:

\[
\begin{align*}
H(\overset{\bigcirc}{\bigcirc}) &\cong H(\overset{\uparrow}{\bigcirc}) \{1\} \oplus H(\overset{\uparrow}{\bigcirc}) \{-1\}
\end{align*}
\]
This categorifies Kuperberg's removing a digon face relation.

To check the square removal relation:

\[
\begin{array}{c}
\quad = \quad + \\
\end{array}
\]

we need to set up maps as above

The maps are the most obvious foams that remove singular points in pairs:

For instance:
Rotating the picture by $90^\circ$ gives the other cobordism between

![Diagram](image)

One checks that these give an orthogonal decomposition of the identity morphism of the square web, which implies that

$$H(\ ] = H(\ ] \oplus H(\ )$$

**Rmk:** A better way to think about these relations is to take the universal additive category generated by these pictures, in which we have

![Diagram](image)

Again, since any web contains either circles, digons or squares, we can inductively reduce $H(\Gamma')$ to $H(\bigcirc \bigcirc \cdots \bigcirc)^{n}$ for any web $\Gamma'$.

**Cor. 3.** $H(\Gamma')$ is a graded free abelian group.

**Pf:** This follows from the circle case and inductively,

$$H(\Gamma \cup \Gamma') \cong H(\Gamma') \otimes H(\Gamma')$$
Rmk: Our construction doesn't give a preferred basis of $H(\Gamma')$.

**$\mathcal{U}_3$ link homology**

After the above effort of categorifying Kuperberg's spider calculus, we are now ready to construct the $\mathcal{U}_3$ link homology.

Now to the resolution of crossings, we apply:

\[
\begin{align*}
\begin{array}{ccc}
\otimes & : & 0 \longrightarrow H(\ ) \{2\} \longrightarrow H(\ ) \{3\} \longrightarrow 0 \\
& & \text{Homological deg 0}
\end{array}
\end{align*}
\]

Then, for any link diagram $D$ with $n$ crossings, we take its complete resolution to obtain $2^n$ webs, apply $H$, and obtain an $n$-dim'l cube of commutative diagrams of free abelian groups, just as we did for the Jones polynomial. The commutativity of the diagrams follows as before, since different paths of maps in the cube come from different orders of composing far apart foams, which are isotopic. It follows that if we adopt the sign convention of tensor products of complexes, the $n$-dim'l cube diagram becomes a multi-complex. We further collapse the multi-grading into a single grading. The total complex so obtained
is our definition of $H(D)$.  

E.g. The Hopf link diagram

The complete resolution:

Thus $H(D)$ is the complex

$$0 \rightarrow H(D) \xrightarrow{(\alpha,\beta)} H(D\Theta^2) \xrightarrow{(-\delta)} H(D) \rightarrow 0$$

Thm 4.

(i). If $D_1$ and $D_2$ are related by a Reidemeister move,

$H(D_1) \cong H(D_2)$
in $\text{Comp}_*\text{gr. Ab}$, the homotopy category of complexes of graded abelian groups.

(2). $H(D)$ is a link invariant.

(3). $\chi(H(L)) = P(L)$.

Sketch of proof:

(2) follows from (1), and (3) is by construction.

The proof of (1) is similar as for the Jones polynomial case. A slight complication comes from different orientations of $R$-moves. For instance, we get different resolutions for $\text{RII}$ moves with different orientations:

$$
\begin{align*}
\left(\begin{array}{c}
\includegraphics[width=1cm]{orientation.pdf}
\end{array}\right) & = \left(\begin{array}{c}
\includegraphics[width=1cm]{orientation1.pdf}
\end{array}\right) \\
\left(\begin{array}{c}
\includegraphics[width=1cm]{orientation2.pdf}
\end{array}\right) & = \left(\begin{array}{c}
\includegraphics[width=1cm]{orientation3.pdf}
\end{array}\right)
\end{align*}
$$

We take the last one as an illustration. The complete resolution of the l.h.s. is:

$$
\begin{align*}
\left(\begin{array}{c}
\includegraphics[width=1cm]{resolution1.pdf}
\end{array}\right) & \rightarrow \left(\begin{array}{c}
\includegraphics[width=1cm]{resolution2.pdf}
\end{array}\right) \\
\left(\begin{array}{c}
\includegraphics[width=1cm]{resolution3.pdf}
\end{array}\right) & \downarrow \quad \downarrow \\
\left(\begin{array}{c}
\includegraphics[width=1cm]{resolution4.pdf}
\end{array}\right) & \rightarrow \left(\begin{array}{c}
\includegraphics[width=1cm]{resolution5.pdf}
\end{array}\right)
\end{align*}
$$

so that the total complex is:
\[ 0 \rightarrow H(\bar{\mathcal{T}}) \rightarrow H(\bar{\mathcal{T}}) \oplus H(\mathcal{O}) \rightarrow H(\mathcal{O}) \rightarrow 0 \]

But
\[ H(\bar{\mathcal{T}}) \cong H(\mathcal{O}) \cong H(\mathcal{O})^{\{1\}} \oplus H(\mathcal{O})^{\{-1\}} \]

\[ H(\mathcal{O}) \cong H(\mathcal{O})^{\{2\}} \oplus H(\mathcal{O}) \oplus H(\mathcal{O})^{\{-2\}} \]

One checks that all (shifted) terms \( H(\mathcal{O}) \) cancel out and we are left with
\[ H(\mathcal{O})^{\{2\}} \]
as desired.

The \( \text{RI} \text{III} \) move need only be checked for one particular orientation once the invariance under \( \text{RI} \) and \( \text{RII} \) is established, since other \( \text{RI} \text{III} \) moves are equivalent to any chosen one modulo \( \text{RI} \) and \( \text{RII} \) moves. We will sketch the following one:

\[ \begin{array}{c}
\begin{array}{c}
\text{II} \\
\text{III}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{II} \\
\text{III}
\end{array}
\end{array} \]
In fact, both sides, after taking complete resolution, give the same total complex (with $H$ omitted):

\[ 0 \rightarrow \varepsilon \rightarrow \omega \rightarrow 0 \]

The invariance follows. \qed

**Extension to tangles**

In this subsection, we sketch the extension of $A_3$ homology theory to the 2 category of tangles. This is in analogy with the $A_2$ case (the Jones polynomial).

First of all we introduce the analogue of crossingless matchings. We will denote by $\varepsilon$ any sequence of signs $\pm$, for instance, $\varepsilon = (+ - +)$. For any such an $\varepsilon$, we define the set $\mathcal{B}^\varepsilon$:

\[ \mathcal{B}^\varepsilon = \{ \text{non-elliptic webs with boundary } \varepsilon \} \]

E.g., $\varepsilon = (+ - +)$

\[ \mathcal{B}^\varepsilon = \left\{ \begin{array}{c}
\begin{array}{c}
\xrightarrow{+} \xleftarrow{-} \\
\xleftarrow{+} \xrightarrow{-}
\end{array}
\end{array}, \begin{array}{c}
\begin{array}{c}
\xrightarrow{+} \xleftarrow{-} \\
\xleftarrow{+} \xrightarrow{-}
\end{array}
\end{array} \right\} \]
Notice that $B^\varepsilon$ gives a basis of $U_3 (U_q(U_3))$-invariants of the module $V^{\otimes \varepsilon}$ (viewed as $U_3$-module maps $C \to V^{\otimes \varepsilon}$). $B^\varepsilon = \phi$, unless $3l = \#(\cdot) - \#(\cdot)$, since this holds for the basic building blocks:

Now for any such $\varepsilon$, define $H^\varepsilon$ by

$$H^\varepsilon \equiv \bigoplus_{a,b} B^\varepsilon \cdot H(W(b) \cdot a),$$

where $W$ denotes reflecting webs about the horizontal axis and reversing orientations on arcs.

and $H$ denotes our previous evaluation of closed webs.

Again, we set

$$H(W(d) \cdot C) \otimes H(W(b) \cdot a) \longrightarrow S_{b,c} \cdot H(W(d) \cdot a)$$

as in the $U_2$ case, and when $c = b$, the product is given by evaluating the canonical cobordism from $W(d) \cdot b \sqcup W(b) \cdot a$ to $W(d) \cdot a$. 
The product is associative for the same reason as in the \( \mathfrak{U}_2 \) case. The unit is given by \( 1 = \sum a \in \mathcal{E} \, 1_a \), where \( 1_a \in H(W(a) \mathfrak{a}) \) is given by the canonical cobordism from \( \phi \) to \( W(a) \mathfrak{a} \):

Moreover, to morphisms between \( \mathcal{E} \) and \( \mathcal{E}' \), i.e., webs \( T_{\mathcal{E}'} \) with boundaries \( \mathcal{E}' \sqcup \mathcal{E} \), we assign the bimodule

\[
H(T_{\mathcal{E}'}) \cong \bigoplus_{b \in \mathcal{E}} a \in \mathcal{E} \, H(W(a) T_{\mathcal{E}' b})
\]
It's readily seen that if $T_1^{E'}$, $T_2^{E'}$ can be composed,

$$H(T_1^{E'}; T_2^{E'}) = H(T_1) \otimes_{H^{E'}} H(T_2)$$

Next, to foams $S$ as cobordisms between webs $T_1^{E'}$, $T_2^{E'}$, we assign the morphism of $(H^{E'}, H^E)$ bimodules $H(S)$, defined by closing up $S$ in all possible ways $w(a) \times [0,1], b \times [0,1]$, $a \in B^{E'}, b \in B^E$:

$$H(S) \triangleleft \bigoplus_{a \in B^{E'}, b \in B^E} H(w(a) \times [0,1] \cdot S \cdot b \times [0,1]) : H(T_1) \to H(T_2)$$

So far, this assignment gives us a 2-functor from the 2-category of webs and foams inside $\mathbb{R}^3$ to the 2-category of bimodules and bimodule homomorphisms (compare with flat tangles in the $\mathcal{U}_2$ case).

Finally, this 2-functor extends to the 2 category of tangle cobordisms to the 2 category of $\text{Compl} (\text{bimodules})$ as we did in the $\mathcal{U}_2$ case. The only difference being that matchings / flat tangles / $F$ are now replaced by $B^{E}/\text{webs} / H$. 
Rmk: The rings $H^e$ are expected to be related to various parabolic categories $O$ of $U_3$, or the cohomology rings of some Springer varieties of $U_3$.

**Comparison with $U_2$ case**

In the $U_2$ case (the Jones polynomial), we defined the projective modules for any $a \in B_n$:

$$P_a = \bigoplus_{b \in B_n} F(w(b)a)$$

These projective modules were shown to be indecomposable, and $\text{End}_{H^n}(P_a) \cong A^\otimes N$ for some $N$.

Now in our $U_3$ theory, one only has indecomposability of $P_a$ where

$$P_a = H^e \cdot 1a = \bigoplus_{b \in B^e} H(w(b)a)$$

when $e$ is short enough. For instance, the first case when $P_a$ is decomposable occurs when $e = (++++++--------)$.
In fact, there is a canonical cobordism taking this web to \( b \):

\[
\begin{array}{c}
\text{a} \\
\end{array}
\begin{array}{c}
\text{b}
\end{array}
\]

Denote the composite cobordism from \( a \) to \( b \) above by \( S \), and \( S^* \) the same cobordism foam viewed backwards from \( b \) to \( a \). Then one get

\[
P_a \xleftrightarrow{H(S)} H(S^*) \rightarrow P_b
\]

One checks that \( H(S) \circ H(S^*) = \pm \text{Id}_{P_b} \), so that \( P_b \) is a direct summand of \( P_a \). Hence \( P_a \) is not indecomposable. However, \( P_b \) is indecomposable since

\[\text{End}_{H^e}(P_b) \cong (k[x]/(x^3))^\otimes 6\]

is a local ring.

Another interesting phenomenon that occurs here but not in the \( U_2 \) theory is that different ordering of sequences may give rise to rings that are not Morita equivalent. For instance, consider \( \varepsilon' = (+-+-) \) which is a different ordering of \( \varepsilon = (+-+-) \). We have

\[
B^{\varepsilon'} = \left\{ \begin{array}{c}
\begin{array}{c}
\text{a'}
\end{array}
\begin{array}{c}
\text{b'}
\end{array}
\end{array} \right\}
\]
In $\mathcal{H}^\mathcal{E}$-mod,
\[
\dim \text{End}_{\mathcal{H}^\mathcal{E}}(P\alpha) = \dim \mathcal{H}(\omega(\alpha)\alpha) = \dim \mathcal{H}(\includegraphics[width=0.1\textwidth]{circle.png}) = [2]^2 \cdot [3]_{|g=1} = 12.
\]

But $\dim \text{End}_{\mathcal{H}^\mathcal{E}'}(P\alpha')$ (resp. $b'$) = 9. It follows that these rings are not Morita equivalent.

However, these rings are derived Morita equivalent, in the sense that the complex of bimodules $\mathcal{H}^\mathcal{E} \mathcal{H}(\mathcal{T}) \mathcal{H}^\mathcal{E}'$, where $\mathcal{T}$ is the tangle:
\[
\includegraphics[width=0.5\textwidth]{tangle.png}
\]
gives rise to an invertible map:
\[
F(\mathcal{T}) \otimes_{\mathcal{H}^{++}} (-) : \text{Comp}(\mathcal{H}^{+-}) \longrightarrow \text{Comp}(\mathcal{H}^{+-})
\]
between the homotopy categories of complexes of projective modules over these rings. It's invertible since $\mathcal{T}$ is invertible with inverse:
\[
\includegraphics[width=0.5\textwidth]{tangle_inverse.png}
\]
\[ H(T) \circ H(T') = H(T \circ T') \]
\[ = H( \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{array} ) \]
\[ = H( \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{array} ) \]
\[ = \text{Id} \circ \text{Comp}(H^{++}) \].

and similarly \( H(T') \circ H(T) = \text{Id} \circ \text{Comp}(H^{+-}) \).

We will give a brief recap of (derived) Monita theory in the next section.

Open problems about \( \mathcal{H}_3 \) link homology

1. There is no computer program to compute it at the moment.

2. The analogue of Rasmussen's invariant.
   In the works of Scott - Nieh, Vaz - Mackaay, they defined deformed versions of \( H(\mathcal{I}) \) for webs \( \mathcal{I} \) using equivariant cohomology of \( \mathbb{CP}^2 \), \( F_3 \).
<table>
<thead>
<tr>
<th></th>
<th>$H$</th>
<th>$\tilde{H}$</th>
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<tbody>
<tr>
<td>$\varnothing$</td>
<td>$\mathbb{Z}$</td>
<td>$H^<em>(\text{pt.}, \mathbb{Z}) \cong H^</em>(BU(3), \mathbb{Z})$ [\cong \mathbb{Z}[t_1, t_2, t_3], \text{ deg } t_i = 2i]</td>
</tr>
<tr>
<td>$\circ$</td>
<td>$H^*(\mathbb{C}P^2)$</td>
<td>$H^*(\mathbb{C}P^2, \mathbb{Z}) \cong \mathbb{Z}[x, t_1, t_2, t_3]$ [\cong \frac{\mathbb{Z}[x, t_1, t_2, t_3]}{(x^3 - t_1 x^2 - t_2 x - t_3)}]</td>
</tr>
<tr>
<td>$\circ$</td>
<td>$H^*(F(3))$</td>
<td>$H^<em>(F(3), \mathbb{Z}) \cong H^</em>_{\mathbb{H}}(\text{pt.}, \mathbb{Z})$ [\cong \mathbb{Z}[x_1, x_2, x_3]] [t_1 = x_1 x_2 x_3, t_2 = x_1 x_3 + x_2 x_3 + x_3 x_3]</td>
</tr>
</tbody>
</table>

One defines chain complexes of link diagrams similarly as for $H$. For instance, this assignment satisfies, for any link diagrams $K, L$

$$C(K \cup L) = C(K) \otimes_{\tilde{H}(\varnothing)} C(L).$$

The analogue of Rasmussen's invariant is defined by first setting $t_1, t_2 = 0$. Then,

$$\tilde{H}(K) = (t - \text{torsion}) \oplus \tilde{H}(K)^{\text{free}}$$

and

$$\tilde{H}(K)^{\text{free}} \cong \tilde{H}(\circ) \{-2S'(K)\}$$

There are examples where $S'(K)$ is not proportional to $S(K)$. However, it works equally as well as $S(K)$ to prove Milnor's conjecture.

A problem about $S'(K)$ is to determine which diagrams are adequate for $\tilde{H}$. 