§12. Categorification of the Heisenberg Algebra

Biadjoint functors from finite groups

Let $k$ be a field, $G$ a finite group, and $H$ a subgroup of $G$. We have inclusion of group algebras $k[H] \subseteq k[G]$, and thus adjoint functors:

$$\text{Ind}_H^G \to \text{Res}_H^G \to \text{Coind}_H^G.$$ 

Moreover, under the finiteness assumption (this can be weakened to $[G:H]<\infty$), $k[H] \subseteq k[G]$ is a Frobenius extension, so that $\text{Ind}_H^G \cong \text{Coind}_H^G$, and $\text{Ind}_H^G$ is biadjoint to $\text{Res}_H^G$. We will use the notation $hGg = k[H] \otimes_{k[H]} k[G]$, $G\cdot G = k[G] \otimes_{k[H]} k[G]$ etc.

Biadjointness allows us to apply string notation introduced in §7. We will denote $\text{Ind}_H^G / \text{Res}_H^G$ resp. by:

$$\text{Ind}_H^G : \begin{array}{c} G \leftrightarrow H \\ + \leftrightarrow - \end{array} \quad \quad \text{Res}_H^G : \begin{array}{c} H \leftrightarrow G \\ - \leftrightarrow + \end{array}$$

The biadjointness of them, are then given by oriented cups and caps:

$$\begin{array}{c}
\begin{array}{c}
\text{H} \\
\text{G}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{G} \\
\text{H}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\text{H} \\
\text{G}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{G} \\
\text{H}
\end{array}
\end{array}$$

which are given explicitly by maps of bimodules:
The third map is well-defined since if \( \{g_i'\} \) is another set of representatives, then \( g_i' = h_i g_i \) for some \( h_i \in H \), so that
\[
\sum_{i=1}^{n} g_i'^{-1} \otimes g'_i = \sum_{i} g_i'^{-1} h_i h_i^{-1} \otimes h_i g_i = \sum_{i} g_i'^{-1} \otimes g_i.
\]
Furthermore, it is a \((G,G)\)-bimodule map because \( \forall g \in G, g \cdot g_i g_{\varphi(i)} = g_{\varphi(i)} g_{\varphi(i)} \) where \( \varphi \) is a permutation of \( \{g_1, \ldots, g_n\} \) and \( h_{\varphi(i)} \in H \). Thus
\[
( \sum_{i=1}^{n} g_i^{-1} \otimes g_i ) \cdot g = \sum_{i=1}^{n} g_i^{-1} \otimes h_{\varphi(i)} g_{\varphi(i)}
= \sum_{i=1}^{n} g_i'^{-1} h_{\varphi(i)} \otimes g_{\varphi(i)}
= \sum_{i=1}^{n} g g_{\varphi(i)}^{-1} \otimes g_{\varphi(i)}
= g \left( \sum_{i=1}^{n} g_i^{-1} \otimes g_i \right).
\]
The general theory in §7 gives us:

Thm. \( \text{Ind}^G_H, \text{Res}^H_K \) are acyclic biadjoint functors under these maps.

These maps gives us the relations:

\[
\begin{align*}
\text{Ind}^G_H H &= H \\
\text{Res}^H_K G &= [G:H] G
\end{align*}
\]

Now assume that \( H \subseteq K \subseteq G \) are two subgroups. We depict by an orientend trivalent vertex the natural transformation:

\[
\text{Ind}^G_H \sim \text{Ind}^K_K \text{Ind}^G_H, \quad \text{Ind}^G_H \sim \text{Ind}^H_H
\]

Similarly, for \( \text{Res}^K_K \),

\[
\text{Res}^H_K \sim \text{Res}^K_K \text{Res}^H_K, \quad \text{Res}^K_K \sim \text{Res}^K_K
\]

Then we have:

\[
\begin{align*}
\text{Ind}^G_H H &= G H \\
\text{Res}^H_K G &= G K H
\end{align*}
\]
Next, we will give a graphical interpretation of Mackey’s formula. Let $K \subseteq G \supseteq H$ be finite subgroups of $G$, we can decompose $G$ into $(K, H)$-double cosets $G = \bigsqcup_{i \in I} K g_i H$, so that we have decomposition of $(l^0K, l^0H)$-bimodules:

$$K l^0K G H \equiv \bigoplus_{i \in I} K l^0K g_i H H .$$

Each $l^0K g_i H$ has a simple description as a $(l^0K, l^0H)$-bimodule:

$$l^0K \otimes l^0H \longrightarrow l^0K g_i H$$

$$k \otimes h \longmapsto k g_i h$$

and $k g_i h = k' g_i h' \iff k'' k' = g_i h h'' g_i^{-1} \in K \cap g_i H g_i^{-1} \cong L_i$, so that we have an isomorphism of bimodules:

$$l^0K \otimes_{l^0K[i]} l^0H \cong l^0K g_i H$$

where $L_i$ acts on $K$ by right multiplication, and on $H$ by a $g_i$-twist:

$$\ell \cdot h = (g_i^{-1} \ell g_i) h$$

Then Mackey’s formula follows:

$$Res^G_K \circ Ind^G_H = K l^0K G H \otimes l^0H -$$

$$= \bigoplus_{i \in I} K l^0K g_i H H \otimes l^0H -$$

$$= \bigoplus_{i \in I} K l^0K \otimes_{l^0K[i]} l^0H \otimes l^0H -$$

$$= \bigoplus_{i \in I} Ind^{L_i}_i \circ Res^{H}_i .$$

It acquires the following graphical interpretation:

$$\kappa \quad G \quad H = \sum_{i \in I} \kappa$$

where the label $g_i$ on an arrow indicates the isomorphism of $Ind^G_i$ or $Res^G_i$, coming from inclusions $L_i \hookrightarrow G$ and $g_i^{-1} L_i g_i \hookrightarrow G$. 
Note that the diagrams below come from \((K,H)\)-bimodule maps:

\[
\begin{align*}
\text{Problem: Investigate Mackey's theorem for systems of groups that naturally appear in geometry and number theory, such as } \text{Gal}(\mathbb{Q}/\mathbb{Q}) \text{ or } \pi_1(\text{Manifolds}).
\end{align*}
\]

\[
\textbf{Symmetric groups}
\]

Now we will apply our general theory above to the special case of symmetric groups.

We will further simplify our notation, \(\text{Res}_{S_n}^{S_{n+k}}\), \(\text{Ind}_{S_n}^{S_{n+k}}\) to \(R_{n+k}^n\), \(I_{n+k}^n\) and \((\text{bi-})\text{modules}\) \(\mathbb{S}_n(\mathbb{S}_m) \mathbb{S}_k\) to \(n(m)\mathbb{R}\), etc.

In §5, we have shown for Nil-Coxeter ring that

\[
\text{NC}_n(\text{NC}_{n+1})_{\text{NC}_n} \cong \text{NC}_n \oplus (\text{NC}_n \otimes_{\text{NC}_{n+1}} \text{NC}_n)
\]

which in turn gives us
\[ \text{Res}_{n+1} \circ \text{Ind}_n \cong \text{Id}_n \oplus \text{Ind}_{n-1} \circ \text{Res}_n \]

The proof there is just a version of Mackey's formula, and we only needed the "R III" relation:

\[ \begin{array}{c}
  \begin{array}{c}
    i \\
    i+1 \\
    i+2
  \end{array} \\
  \begin{array}{c}
    j \\
    j+1 \\
    j+2
  \end{array}
\end{array} = \begin{array}{c}
  \begin{array}{c}
    i \\
    i+1 \\
    i+2
  \end{array} \\
  \begin{array}{c}
    j \\
    j+1 \\
    j+2
  \end{array}
\end{array} \]

to obtain the decomposition of bimodules above. Thus for \( \mathbb{I} \mathbb{K} [\mathbb{S}_n] \), we also have:

\[ n(n+1)n \cong n(n)n \oplus (n_{n-1}n) \]

so that we also have:

\[ R_{n+1}^n \circ I_n^{n_1} \cong \text{Id}_n \oplus I_{n-1}^n \circ R_n^{n_1}. \]

Next, notice that the functor \( I^2 = I_{n+2}^{n+2} \circ I_n^{n_1} \) admits an endomorphism coming from the \((n+2, n_2)\)-bimodule map

\[ \begin{array}{c}
  \begin{array}{c}
    n+2(n+2)n \\
    n+2(n+2)n
  \end{array} \\
  \begin{array}{c}
    g_{\mathbb{S}_{n+1}} \\
    g
  \end{array}
\end{array} \]

where \( \mathbb{S}_{n+1} = (n+1, n+2) \). We will denote this endomorphism by:

\[ \begin{array}{c}
  \begin{array}{c}
    n+2 \\
    n+1
  \end{array} \\
  \begin{array}{c}
    n \\
    1
  \end{array}
\end{array} \]

Then we have the relations:
\[
\begin{align*}
\leftarrow & = \uparrow \uparrow \uparrow, \\
\rightarrow & = \rightarrow \rightarrow,
\end{align*}
\]

which follows from the corresponding relations in $S_n$. Similarly, $R^2 = R_{n+2} \circ R_n$ has as an endomorphism $S_{n+1}$:

\[
\begin{array}{ccc}
\uparrow & \uparrow \\
\downarrow & \downarrow \\
\uparrow & \uparrow \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{c}
n(n+2)_{n+2} \\
S_{n+1} \\
g \\
\emptyset \\
n(n+2)_{n+2}
\end{array}
\]

which we depict as:

\[
\begin{array}{cccc}
R & n+1 & R \\
n & \times & n+2 \\
R & n+1 & R
\end{array}
\]

Using cups and caps we can produce crossings between $R$ and $I$ as well:

\[
\begin{array}{cc}
\begin{array}{ccc}
R & I \\
I & R \\
\end{array} & \cong \\
\begin{array}{ccc}
R & I \\
I & R \\
\end{array}
\end{array}
\]

etc.

By the discussion above, we have:

\[
\begin{array}{c}
\downarrow \uparrow = \leftarrow \leftarrow + \\
\end{array}
\]

which is encoded in $RI \cong IR \oplus \text{Id}$ as follows:
and the relations (exercise):

\[ \uparrow \downarrow = \begin{array}{c} \otimes \end{array} , \quad \begin{array}{c} \otimes \end{array} = 1 , \quad \begin{array}{c} \otimes \end{array} = 0 . \]

These are relations that do not depend on \( n \). Some relations, however, do depend on \( n \). For instance,

\[ \otimes^{-1} n = n \cdot n \]

The monoidal category \( \mathcal{H} \)

Now we define an abstract monoidal \( \text{k-linear} \) category \( \mathcal{H'} \).

Objects of \( \mathcal{H'} \) are defined to be finite direct sums of tensor products of \( I \) or \( R \):

\[ \begin{array}{c} I \quad R \quad I \quad I \quad R \end{array} \]

Morphisms of \( \mathcal{H'} \) between objects are \( \text{k-linear} \) combinations of oriented string diagrams, with at most simple crossings:
The morphisms are required to satisfy isotopies relative to boundaries and local relations modeled on those relations above for symmetric groups which do not depend on $n$:

\[
\begin{align*}
\uparrow \downarrow &= = \quad , \\
\downarrow \uparrow &= \quad + \\
\&= \quad , \\
\bigcirc &= \quad , \\
\bigcirc &= \quad .
\end{align*}
\]

One can check that these relations imply:

\[
\begin{align*}
\uparrow \bigcirc &= \quad \\
\bigcirc &= \quad
\end{align*}
\]

which further implies that

\[
\begin{align*}
\uparrow \bigcirc &= \quad \\
\bigcirc &= \quad
\end{align*}
\]

holds with arbitrary orientation.
The right curl doesn’t simplify, and it will be convenient to relabel it as a dot:

\[ \uparrow \equiv \circ \]

Moreover, clockwise circles carrying dots do not simplify:

\[ \kbullet \circ \equiv \kbullet \circ \cdot \circ \cdot \circ \]

The dot is related to the Jucy–Murphy elements in \( l_k[S_n] \) as follows.

Lemma. Fix an \( n \in \mathbb{N} \), we have:

\[ n+1 \uparrow \circ n = \sum_{i=1}^{n} (i, n+1) \equiv J_n. \]

the \( n \)-th Jucy–Murphy elements.

Pf: An easy computation by decomposing the right curl into a cup, a crossing, and a cap:

\[ \uparrow \circ \cdot \circ \cdot \circ \]

We recall that Jucy–Murphy elements commute with each other. For each \( n \in \mathbb{N}, J_0 = 0, J_1 = (12), \ldots, J_{n-1} \) form a maximal commutative subalgebra of \( l_k[S_n] \) if \( \text{char} k = 0 \). The commutativity of these
elements now becomes planar isotopy relations:

\[
\begin{array}{c}
\uparrow \uparrow \downarrow \downarrow = \uparrow \uparrow \uparrow \downarrow \\
\end{array}
\]


The following lemma is an easy consequence of the defining relations of \( \mathcal{H}' \):

Lemma. In \( \mathcal{H}' \), we have,

\[
\begin{array}{c}
\begin{array}{c}
\times = \times + \uparrow \uparrow \\
\times = \times + \uparrow \uparrow \\
\end{array}
\end{array}
\]

This relation, together with all the upward pointing relations, reminds us of the notion of the degenerate affine Hecke algebra:

Def. (Degenerate AHA on \( n \)-strands \( \text{DH}_n \)). \( \text{DH}_n \) is the \( lk \)-linear diagrammatic algebra on \( n \) strands (like \( \text{NC}_n \)) carrying dots subject to the local relations:

\[
\begin{array}{c}
\end{array}
\]
Notice that \( \mathbb{I}[S_n] \) naturally embeds into \( \mathcal{D}H_n \) as a subalgebra. Moreover, we have a retraction by assigning a dot on the \( k \)-th strand the \( k \)-th Jucy-Murphy element:

\[
\begin{align*}
\mathcal{D}H_n & \longrightarrow \mathbb{I}[S_n] \\
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} & \begin{array}{c}
\bullet \\
\circ \\
\circ \\
\end{array} & \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\text{J}_k \\
\end{align*}
\]

Notice that

\[
\begin{array}{c}
\vdots \\
\circ \\
\end{array} \longrightarrow \begin{array}{c}
\circ \\
\text{J}_k \\
\end{array} \longrightarrow \circ \in \mathbb{I}[S_n].
\]

Now we will state the first main result about \( \mathcal{H}' \). Before that, we state the following lemma, which follows from an induction argument.
Lemma. The counter-clockwise circle carrying dots can be reduced to polynomial combinations of clockwise circles with dots on them.

For instance,

$$\begin{array}{c}
\text{4-circled} \quad = \quad \text{2-circled} \quad + \quad \text{circles}
\end{array}$$

Thm. (1). $\text{End}_\mathcal{H}(1_\mathcal{H}) \cong \mathbb{I}_k[C_0, C_1, C_2, \ldots]$, where

$$C_0 = \begin{array}{c}
\text{circled}
\end{array}.$$}

In other words, this is saying that any closed diagram in $\mathcal{H}$ can be reduced to linear combinations of pictures with only clockwise circles with dots.

(2). $\text{End}_\mathcal{H}(I^n) \cong \text{DH}_n \otimes \text{End}_\mathcal{H}(1_\mathcal{H})$.

In other words, any pictures going from $n$ bottom $I$'s to $n$ top $I$'s can be reduced to linear combinations of elements of $\text{DH}_n$ with some circles attached on their right hand side.

In other words, this is saying that any closed diagram in $\mathcal{H}$ can be reduced to linear combinations of pictures with only clockwise circles with dots.
Notice that by turning everything above upside-down we get the analogous results for $\mathbb{R}^n$.

For the proof, see M. Khovanov, Heisenberg algebra and a graphical calculus.

Def. The category $\mathcal{H}$ is defined to be the Karoubi envelope of $\mathcal{H}'$.

$\mathcal{H}$ is a $k$-linear, monoidal category since $\mathcal{H}'$ is. By the thm above, $\mathbb{C}[S_n] \otimes \mathbb{C}Hn$ acts on $\mathbb{C}^n (\mathbb{R}^n) \in \text{Ob}(\mathcal{H}')$. Let $e^+_n (e^-_n)$ be the complete symmetrizer (anti-symmetrizer) in $\mathbb{C}[S_n]$ (char $k = 0$), and define $A_n \triangleq (\mathbb{C}^n, e^+_n)$, $B_n = (\mathbb{R}^n, e^-_n) \in \text{Ob}(\mathcal{H}')$.

Prop. We have in $\mathcal{H}$ that

1. $A_0 = B_0 = \text{Id}$, $A_n = B_n = 0$ if $n < 0$.
2. $RI \cong IR \oplus \text{Id}$
3. $A_n A_m \cong A_m A_n$, $B_n B_m \cong B_m B_n$
4. $B_m A_n \cong A_n B_m \oplus A_{n-1} B_{m-1}$

Hence $K_0(\mathcal{H})$ is a ring, in which

$[B_m], [A_n] = [A_n][B_m] + [A_{n-1}][B_{m-1}]$

These elements $[A_n], [B_n]$ can be shown to generate the Heisenberg algebra

$\mathcal{H} = \mathbb{C} \langle P_n, Q_n \rangle_{n \geq 0} / (P_n P_m = P_m P_n, Q_n Q_m = Q_m Q_n, P_n Q_m = Q_m P_n + \delta_{nm})$

Thm. There is an injection

$\psi: \mathcal{H} \longrightarrow K_0(\mathcal{H})$
Conjecture: $\phi$ is an isomorphism.

The proofs can be found in the above mentioned paper.

Rmk: There is another categorification of $H$ by Cautis-Licata, where they essentially used the rings $\text{lk}[S_n] \times T^{n}$, where $T = \Lambda^2 \mathbb{C}^2 \times \text{lk}[G]$, and $G$ is a finite subgroup of $SU(2)$. The algebra $T$ describes the derived categories of coherent sheaves on some Nakajima quiver varieties (Kapranov).