Algebraic topology

Homework 6. Due Monday, October 30.

1. Each topological space X is weakly homotopy equivalent to a CW-complex. Use this result to construct a functor from the category HTop of topological spaces and continuous maps up to homotopy to HCW, the category of CW-complexes and continuous maps between them up to homotopy. Your functor must be right adjoint to the inclusion functor HCW $\subset$ HTop.

2. Let V be a complex of vector spaces over a field k :

$$\cdots \xrightarrow{\partial} V_n \xrightarrow{\partial} V_{n-1} \xrightarrow{\partial} \ldots$$

Show that V is isomorphic to the direct sum of complexes

$$0 \longrightarrow H_n(V) \longrightarrow 0,$$

with $H_n(V)$ in degree n, and contractible complexes of the form

$$0 \longrightarrow W_n \xrightarrow{\mathrm{Id}} W_n \longrightarrow 0$$

for some collection of vector spaces W_n, with the two W_n above sitting in degrees n and $n-1$, respectively. Conclude that in HCom(k) the complex V is isomorphic to its cohomology complex with the zero differential:

$$\cdots \xrightarrow{0} H_n(V) \xrightarrow{0} H_{n-1}(V) \xrightarrow{0} \ldots$$

3. To a short exact sequence of complexes

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

we assigned a sequence of homology groups

$$\ldots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \ldots$$

Check that this sequence is exact (or read the proof in Hatcher).

Exercises 14, 16 at the end of section 2.1.