Algebraic topology

Discussion 5 Section 4.2 of Hatcher (subsections on the Freudenthal theorem, Eilenberg-MacLane spaces, fiber bundles).

Homework 5. Due Monday, October 16.

1. Show that the spaces $\mathbb{S}^2$ and $\mathbb{S}^3 \times \mathbb{CP}^\infty$ have isomorphic homotopy groups but are not homotopy equivalent.

2. Let X_n be the subset of $\mathbb{C}^n$ consisting of n-tuples of pairwise distinct complex numbers $(z_1, \dots, z_n)$.

(a) Show that the forgetful map $X_n \longrightarrow X_{n-1}$ given by omitting z_n make X_n a fiber bundle with base X_{n-1}.

(b) Using induction on n and the long exact homotopy sequence of a fibration prove that X_n is a $K(H,1)$-space for some group H. This group is called the pure braid group on n strands.

(c) The symmetric group S_n acts on X_n by permuting the z_i's. Let $X_n \longrightarrow X_n'$ be the quotient map. Explain why this map is a locally-trivial covering, compute its group of deck transformations, and show that X_n' is a $K(G,1)$-space for some group G (called the braid group on n strands).

(d) Show that there exists a short exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{S}_n \longrightarrow 1.$$

(e) Determine the braid group and the pure braid group in the case $n = 2$.

3. Let $p : E \longrightarrow B$ be a Serre fibration, $b \in B$ and $F = p^{-1}(b)$. We constructed a homomorphism $p_* : \pi_n(E,F) \longrightarrow \pi(B)$ and checked its injectivity. Check that the homomorphism is surjective and conclude that p_* is an isomorphism.

4. Suppose that $p : E \longrightarrow B$ is a locally-trivial covering. What does the long exact sequence for p tell us about the homotopy groups of E, B, and F?

5. Let G be a group and X a simply-connected space. Show that for the product $K(G,1) \times X$ the action of π_1 on π_n is trivial for all $n > 1$.