Modern Algebra II, spring 2022

Homework 9, due Wednesday April 6.

1. (20 points) Which of the following numbers are constructible using a ruler and compass? Briefly justify your answer.

\[ \frac{\sqrt{3}}{2}, \sqrt{6 + \sqrt{7}}, \sqrt{5} - 1, \sqrt{2} + 1. \]

2. (20 points) Briefly sketch the steps involved into constructing numbers \( \sqrt{2} \), \( \sqrt{\sqrt{2} + 1} \) and \( \sqrt{\sqrt{\sqrt{2} + \sqrt{3} + 1}} \) using a ruler and compass.

3. (20 points) Suppose we have a ruler and compass, as before, but are given 3 points \( A, B, C \) on a line in the plane, with \( B \) between \( A \) and \( C \) and distances \( |AB| = 1, |BC| = \sqrt{2} \). Explain how to modify the arguments in this week’s lectures to show that \( \sqrt{2} \) is not constructible with these assumptions. (Hint: What are the properties of the tower of fields \( \mathbb{Q} \subset K_0 \subset K_1 \subset \cdots \subset K_n \) where the field \( K_i \) is generated by the coordinates of \( A, B, C \) and of the next \( i \) points that we create? What can you say about the degree \( [K_n : \mathbb{Q}] \)?)

4. (20 points) (a) Recall the definition of a normal extension \( E/F \) (or see our usual references). Explain in your own words what is an obstacle for an extension to be normal.

(b) Let \( E/F \) be a degree two extension. Prove that \( E \) is normal. Hint: pick an element \( \alpha \in E \setminus F \). Write down its irreducible polynomial \( f(x) \). Can you show that \( E \) is a splitting field of \( f(x) \)? You need to check that \( E \) contains all roots of \( f(x) \), not just \( \alpha \).

(c) Look through class notes and find an example of degree 3 extension of \( \mathbb{Q} \) which is not normal. Generalize that example and describe a degree \( n \) extension of \( \mathbb{Q} \) which is not normal, for any \( n \geq 3 \).

(d) Explain why any extension of finite fields \( \mathbb{F}_q \subset \mathbb{F}_{q^n} \) is normal.

5. (20 points) For any automorphism \( \sigma \) of a ring \( R \) we can define the subring \( R^\sigma \) of elements fixed by \( \sigma \).

(a) Give a definition of \( R^\sigma \) using mathematical notations (via sets and quantifiers) and prove that \( R^\sigma \) is a subring.

(b) Suppose \( R = F[x] \), where \( F \) is a field, and \( \sigma \) takes a polynomial \( f(x) \) to \( f(-x) \). For instance, if \( f(x) = a + bx + cx^2 \), then \( \sigma(f) \) is the polynomial \( a - bx + cx^2 \). Prove that the subring \( R^\sigma \) of polynomials invariant under \( \sigma \) (equivalently, fixed by \( \sigma \)) is the subring \( F[x^2] \) if \( \text{char } F \neq 2 \). What happens when \( F \) has characteristic two?