Solutions to HW 6

Karol Koziol

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Saff/Snider, p. 354

1. Consider the integral \( \int_0^\infty \frac{\sqrt{z}}{z^2+1} \, dx \). We define \( \log(z) \) using the branch cut going from 0 to \( \infty \) along the positive real axis. This means \( \log(z) = \log(r) + i\theta, \theta \in [0, 2\pi) \). Thus, \( \sqrt{z} = e^{(\log(r)+i\theta)/2} \), for \( z = re^{i\theta} \). We take as our contour \( \Gamma \) the contour going from \( \epsilon \) to \( \rho \) along the real axis, the circular contour \( C_\rho \) given by \( \rho e^{it}, t \in [0, 2\pi] \), the path going from \( \rho \) to \( \epsilon \), and the circular contour \( C_\epsilon \) given by \( \epsilon e^{-it}, t \in [0, 2\pi] \). As \( z \) approaches the real axis from the “upper side” of the cut, the integral over \([\epsilon, \rho]\) approaches

\[
\int_\epsilon^\rho \frac{\sqrt{x}}{x^2+1} \, dx,
\]

while approaching from the bottom gives

\[
\int_\rho^\epsilon \frac{e^{(\log(r)+2\pi i)/2}}{x^2+1} \, dx = \int_\rho^\epsilon \frac{-\sqrt{x}}{x^2+1} \, dx = \int_\epsilon^\rho \frac{\sqrt{x}}{x^2+1} \, dx.
\]

It remains to show that the integrals along \( C_\rho \) and \( C_\epsilon \) go to 0 as \( \rho \to \infty \) and \( \epsilon \to 0^+ \). We have

\[
\left| \int_{C_\rho} \frac{\sqrt{z}}{z^2+1} \, dz \right| \leq \int_{C_\rho} \frac{\sqrt{\rho}}{\rho^2-1} |dz| = \frac{2\pi \rho^{3/2}}{\rho^2-1}
\]

and

\[
\left| \int_{C_\epsilon} \frac{\sqrt{z}}{z^2+1} \, dz \right| \leq \int_{C_\epsilon} \frac{\sqrt{\epsilon}}{|\epsilon^2-1|} |dz| = \frac{2\pi \epsilon^{3/2}}{|\epsilon^2-1|}.
\]
Both of these terms go to 0 as $\rho \to \infty$ and $\epsilon \to 0^+$, respectively. Putting all of this together gives

$$2 \int_0^\infty \frac{\sqrt{x}}{x^2 + 1} \, dx = \lim_{\rho \to \infty, \epsilon \to 0^+} \int_T \frac{\sqrt{z}}{z^2 + 1} \, dz$$

$$= 2\pi i \left( \text{Res} \left( \frac{\sqrt{z}}{z^2 + 1}; i \right) + \text{Res} \left( \frac{\sqrt{z}}{z^2 + 1}; -i \right) \right)$$

$$= 2\pi i \left( \frac{\sqrt{2} e^{\pi i/4}}{2} + \frac{\sqrt{2} e^{\pi i/4}}{-2i} \right)$$

$$= \frac{2\pi}{\sqrt{2}},$$

which gives the answer.

2. We wish to evaluate $\int_0^\infty \frac{z^{\alpha-1}}{x+1} \, dx$, and we use the same contour as in the previous problem. We now have the function $z^{\alpha-1}$, which is defined, as before, as $z^{\alpha-1} = e^{(\alpha-1)(\log(r)+i\theta)}, \theta \in [0, 2\pi)$. As $z$ approaches the real axis from above, the integral becomes

$$\int_\rho^\infty \frac{z^{\alpha-1}}{x+1} \, dx,$$

while approaching from below gives

$$\int_\epsilon^\rho \frac{e^{(\alpha-1)(\log(r)+2\pi i)}}{x+1} \, dx = -e^{2\pi i (\alpha-1)} \int_\epsilon^\rho \frac{z^{\alpha-1}}{x+1} \, dx.$$

On the contours $C_\rho$ and $C_\epsilon$, we have

$$\left| \int_{C_\rho} \frac{z^{\alpha-1}}{z+1} \, dz \right| \leq \int_{C_\rho} \frac{\rho^{\alpha-1}}{\rho - 1} |dz| = \frac{2\pi \rho^\alpha}{\rho - 1}$$

$$\left| \int_{C_\epsilon} \frac{z^{\alpha-1}}{z+1} \, dz \right| \leq \int_{C_\epsilon} \frac{\epsilon^{\alpha-1}}{\epsilon - 1} |dz| = \frac{2\pi \epsilon^\alpha}{\epsilon - 1}.$$

As before, both of these go to 0 as $\rho \to \infty$ and $\epsilon \to 0^+$, provided $0 < \alpha < 1$. Thus, we have

$$(1 - e^{2\pi i (\alpha-1)}) \int_0^\infty \frac{x^{\alpha-1}}{x+1} \, dx = \lim_{\rho \to \infty, \epsilon \to 0^+} \int_T \frac{z^{\alpha-1}}{z+1} \, dz$$

$$= 2\pi i \text{Res} \left( \frac{z^{\alpha-1}}{z+1}; -1 \right)$$

$$= 2\pi i e^{(\alpha-1)\pi i}.$$
Note that $e^{2\pi i (\alpha - 1)} = e^{2\pi i \alpha}$. Thus, we get

$$
\int_0^\infty \frac{x^{\alpha - 1}}{x + 1} \, dx = \frac{-2\pi i e^{\pi i}}{1 - e^{2\pi i \alpha}} = \frac{-2\pi i}{e^{-\pi i \alpha} - e^{\pi i \alpha}} = \frac{\pi}{\sin(\pi \alpha)}.
$$

Stein/Shakarchi p. 103

3. Let $a > 0$, and consider the integral $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} \, dx$. Note that this integral is simply $\Re(\int_{-\infty}^{\infty} e^{ix} \, dx)$, and we will use this to compute the integral. Note that, if we write $z = x + iy$, we have $|e^{iz}| = e^{\Re(iz)} = e^{-y}$. Therefore, the numerator of the fraction is bounded by 1 in the upper half plane, and we may use the contour $\gamma_1$ going from $-R$ to $R$, followed by the semicircular contour $\gamma_2(t) = Re^{it}, t \in [0, \pi]$. We have an easy bound:

$$
\left| \int_{\gamma_2} \frac{e^{iz}}{z^2 + a^2} \, dz \right| \leq \int_{\gamma_2} \frac{e^{-y}}{R^2 - a^2} \, |dz| \\
\leq \int_{\gamma_2} \frac{1}{R^2 - a^2} \, |dz| \\
= \frac{\pi R}{R^2 - a^2},
$$

and therefore the integral goes to 0 as $R \to \infty$. Using the residue theorem, we have

$$
\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} \, dx = \lim_{R \to \infty} \int_{\gamma_1 \cup \gamma_2} \frac{e^{iz}}{z^2 + a^2} \, dz \\
= 2\pi i \text{Res} \left( \frac{e^{iz}}{z^2 + a^2}; ai \right) \\
= 2\pi i \left( \frac{e^{iz}}{2z} \right)_{z=ai} \\
= \frac{\pi e^{-a}}{a}.
$$

Thus,

$$
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} \, dx = \Re \left( \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} \, dx \right) = \Re \left( \frac{\pi e^{-a}}{a} \right) = \frac{\pi e^{-a}}{a}.
$$

4. We wish to evaluate the integral $\int_{-\infty}^{\infty} \frac{z \sin(x)}{x^2 + a^2} \, dx$, for $a > 0$. Notice that, as in the previous problem, we have that this integral is equal to $\Im(\int_{-\infty}^{\infty} \frac{e^{iz}}{x^2 + a^2} \, dx)$. In this problem we have to take more care about how we choose our contour. Consider the following contour $\Gamma$: $\gamma_1$ is the path on the real axis from $-R$ to $R$, $\gamma_2$ is the straight path from $R$ to $R + i\sqrt{R}$, $\gamma_3$ is the straight path from $R + i\sqrt{R}$ to $-R + i\sqrt{R}$, and $\gamma_4$ is the straight path from $-R + i\sqrt{R}$ to $-R$. We need to bound the function on each of these segments. We have $|ze^{iz}| = |z|e^{-y}$ for $z = x + iy$, and using this, we get, for $R$ sufficiently
large,

\[
\left| \int_{\gamma_2} \frac{ze^{iz}}{z^2 + a^2} \, dz \right| \leq \int_{\gamma_2} \frac{|z|e^{-y}}{|z|^2 - a^2} |dz| \\
\leq \int_{\gamma_2} \frac{\sqrt{R^2 + R}}{R^2 - a^2} |dz| \\
= \frac{\sqrt{R}\sqrt{R^2 + R}}{R^2 - a^2}
\]

As \( R \to \infty \), all of these integrals go to 0 (note that taking the rectangular contour with vertices \(-R, R, -R + iR, R + iR\) would not give enough decay). Therefore, we have

\[
\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} \, dx = \lim_{R \to 0} \int_{\gamma} \frac{ze^{iz}}{z^2 + a^2} \, dz = 2\pi i \text{Res} \left( \frac{ze^{iz}}{z^2 + a^2}; ai \right) = 2\pi i \left( \frac{ze^{iz}}{2z} \right)_{z=ai} = \pi ie^{-a}.
\]

Therefore \( \int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2 + a^2} \, dx = \Im(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} \, dx) = \pi e^{-a} \).

5. We consider the case when \( \xi \leq 0 \). The other case is similar, except we must choose the “other” semicircular contour. Let \( z = x + iy \). Then we have \( |e^{-2\pi iz\xi}| = e^{\Re(-2\pi iz\xi)} = e^{2\pi \xi y} \). Therefore, since \( \xi \) is non-positive, we should choose the semicircular contour \( \gamma(t) = Re^{it}, t \in [0, \pi] \), that lies in the upper
half plane, so that the numerator of the fraction is bounded. We have, for $R$ sufficiently large,

$$\left| \int_{\gamma} \frac{e^{-2\pi i \xi}}{(1 + z^2)^2} \, dz \right| \leq \int_{\gamma} \frac{e^{2\pi y \xi}}{(|z|^2 - 1)^2} \, |dz|$$

$$\leq \int_{\gamma} \frac{1}{(R^2 - 1)^2} \, |dz|$$

$$= \frac{\pi R}{(R^2 - 1)^2}.$$

Therefore the integral on this contour goes to zero as $R$ goes to $\infty$. We now have

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i \xi}}{(1 + x^2)^2} \, dx = \lim_{R \to \infty} \int_{[-R,R] \cup \gamma} \frac{e^{-2\pi i \xi}}{(1 + z^2)^2} \, dz$$

$$= 2\pi \text{Res} \left( \frac{e^{-2\pi i \xi}}{(1 + z^2)^2} ; i \right)$$

$$= 2\pi i \lim_{z \to i} \frac{d}{dz} \left( \frac{e^{-2\pi i \xi}}{(z + i)^2} \right)$$

$$= 2\pi i \lim_{z \to i} \frac{(-2\pi i \xi)e^{-2\pi i \xi}(z + i) - 2e^{-2\pi i \xi}}{(z + i)^3}$$

$$= \frac{2\pi i}{4i}(1 - 2\pi \xi)e^{2\pi \xi}$$

$$= \pi \frac{(1 - 2\pi \xi)e^{2\pi \xi}}{4i}.$$

Note that, for $\xi \leq 0$, we have $|\xi| = -\xi$, so the last equation reads $\frac{\pi}{2}(1 + 2\pi |\xi|)e^{-2\pi |\xi|}$, as desired.

7. We consider the integral $\int_{0}^{2\pi} \frac{d\theta}{(a + \cos(\theta))^2}$, using the same method for trigonometric integrals as in Homework 4. Using the substitution $z = e^{i\theta}$, we have $\cos(\theta) = \frac{z + z^{-1}}{2}$, $d\theta = dz/iz$. Thus

$$\frac{d\theta}{(a + \cos(\theta))^2} = \frac{dz}{iz} \frac{1}{(a + \frac{z + z^{-1}}{2})^2} = \frac{-4izdz}{(z^2 + 2az + 1)^2}.$$

This function has double poles at the points $z = -a \pm \sqrt{a^2 - 1}$, and of these, the only one contained
inside the contour $|z| = 1$ is $z = -a + \sqrt{a^2 - 1}$. Therefore,

$$\int_0^{2\pi} \frac{d\theta}{a + \cos(\theta)} = \int_{|z|=1} \frac{-4izdz}{(z^2 + 2az + 1)^2} = 2\pi i \text{Res} \left( \frac{-4iz}{(z^2 + 2az + 1)^2}; -a + \sqrt{a^2 - 1} \right) = 2\pi i \lim_{z \to -a + \sqrt{a^2 - 1}} \frac{d}{dz} \frac{-4iz}{(z + a + \sqrt{a^2 - 1})} = 2\pi i \frac{-ia}{\sqrt{a^2 - 1}} = \frac{2\pi a}{\sqrt{a^2 - 1}}.$$

8. We use the same method as in the previous problem. We have

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)} = \int_{|z|=1} \frac{dz}{iz} \frac{1}{a + \frac{b}{2}z + \frac{b}{2}z^{-1}} = \frac{-idz}{(b/2)z^2 + az + (b/2)}.$$

This function has simple poles at $z = (\pm a \pm \sqrt{a^2 - b^2})/b$, and the only one of these contained in the contour $|z| = 1$ is $z = (a + \sqrt{a^2 - b^2})/b$. We then have

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)} = \int_{|z|=1} \frac{-idz}{(b/2)z^2 + az + (b/2)} = 2\pi i \text{Res} \left( \frac{-i}{(b/2)z^2 + az + (b/2)}; -a + \sqrt{a^2 - b^2} \right) = 2\pi i \left( \frac{-i}{bz + a} \right)_{z = \frac{-a + \sqrt{a^2 - b^2}}{b}} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

10. In order to evaluate $\int_0^\infty \frac{\log(x)}{x^2 + a^2} dx$, we choose the branch of the logarithm cut along the negative imaginary axis; more precisely, we define $\log(z) = \log(r) + i\theta, \theta \in [-\pi/2, 3\pi/2]$ for $z = re^{i\theta}$. We take our contour $\Gamma$ the one on p. 105. As $z$ approaches the positive real axis, our integral approaches

$$\int_\epsilon^R \frac{\log(x)}{x^2 + a^2} dx,$$

while if $z$ approaches the negative real axis, the integral approaches

$$\int_0^\epsilon \frac{\log(r) + i\pi}{r^2 + a^2} dr = \int_0^R \frac{\log(x) + i\pi}{x^2 + a^2} dx.$$
It remains to show that the integral on the semicircular contours goes to 0 as \( R \to \infty \) and \( \epsilon \to 0^+ \). We have, for \( R \) sufficiently large,

\[
\left| \int_{C_R} \frac{\log(z)}{z^2 + a^2} \, dz \right| \leq \int_{C_R} \frac{\log(|z|) + \pi}{|z|^2 - a^2} \, |dz| = \pi R (\log(R) + \pi) \frac{R}{R^2 - a^2}.
\]

Therefore, both of these go to 0 as \( R \to 0 \) and \( \epsilon \to 0^+ \), respectively. Note also that \( \int_0^\infty \frac{i\pi}{x^2 + a^2} \, dx = \frac{i \pi^2}{2a} \).

Thus, we have

\[
i\frac{\pi^2}{2a} + 2 \int_0^\infty \frac{\log(x)}{x^2 + a^2} \, dx = \lim_{R \to \infty, \epsilon \to 0^+} \int_{C_R} \frac{\log(z)}{z^2 + a^2} \, dz
= 2\pi i \text{Res} \left( \frac{\log(z)}{z^2 + a^2}, ai \right)
= 2\pi i \left( \frac{\log(ai)}{2i} \right)
= 2\pi i \frac{\log(a) + \frac{\pi}{2} i}{2a}
= \pi \log(a) + i \frac{\pi^2}{2a}.
\]

Therefore, we see that \( \int_0^\infty \frac{\log(x)}{x^2 + a^2} \, dx = \frac{\pi \log(a)}{2a} \).