Review: The basic definitions

- Recall that the Weil representation arises as follows. Let $F$ be a local (or a finite, that works too) field of characteristic not 2, and $W$ an $F$ vector space of even dimension with an alternating inner product $\langle , \rangle$.
- $H(W)$ is called the Heisenberg group and as a set it’s equal to $W \oplus F$ but the group law is:
  $$(w_1, f_1)(w_2, f_2) = (w_1 + w_2, f_1 + f_2 + \frac{1}{2} \langle w_1, w_2 \rangle)$$
- If you choose a different pairing, you still get an isomorphic Heisenberg group. Still, keep in mind that a lot of the representations we describe are dependent on $\langle , \rangle$.
- It’s clear that the center of this group is the set $(0, F)$. Any character additive character of $F$ is also a character of the center of $H(W)$.

**Theorem. (Stone-von Neumann)** Given any additive character $\psi$ of $F$ there’s a unique irreducible representation $(\rho_\psi, S)$ of $H(W)$ with central character $\psi$.

- The group $Sp(W)$ acts on $W$ and so it also acts on $H(W)$ by $g(w, t) = (w^g, t)$.
- If we consider $g \in Sp(W)$, the representation $\rho_\psi^g(h) = \rho_\psi(h^g)$ again has central character $\psi$, and it is therefore isomorphic to $\rho_\psi$.
- Let $A$ be the linear map, an intertwiner $A(g) : S \to S$ that satisfies
  $$A(g) \rho_\psi(h) A(g)^{-1} = \rho_\psi^g(h)$$
- This almost gives us a representation of $Sp(W)$, but since $A(g)$ is only defined up to scalar, we actually get a “projective” representation, a map $\omega_\psi : Sp(W) \to PGL(S)$.
- We spent a lot of time fixing this problem, raising this representation to a cover of $Sp(W)$ to make it a true representation. When you try to make the representation well-defined so you choose a lift $\omega'$ you get a certain cohomology class given by $\omega'(g_1, g_2) = c(g_1, g_2) \omega'(g_1) \omega'(g_2)$, and this actually specifies a cover of $Sp(W)$.

Review: The Cohomology Cycle

- What Dan showed last time was that this cohomology class $c \in H^2(Sp(W), \mathbb{C}^\times)$ which classifies $\mathbb{C}^\times$-extensions is an element of $H^2(Sp(W), \mu_2)$,
• the point is that you only need to lift it to a \( \mu_2 \)-extension, or, double cover of \( Sp(W) \) to make the representation defined.
• This double cover is called the Metaplectic group. It doesn’t depend on \( \psi \).
• Furthermore, he showed that the cocycle is not trivial, so it’s not a split extension. So that the representation \( g \mapsto A(g) \) is never well-defined.

**Models: Introduction**

• Now we want to understand the representation. Once you know the central character you can start thinking about how the representation is induced from some one-dimensional representation.
• Indeed, if \( Y \subset W \) is an isotropic space, meaning that \( <y_1, y_2> = 0 \) for all \( y_1, y_2 \in Y \), and \( Y \) is maximal wrt this property,
• then we can form \( H(Y) \subset H(W) \) and now, \( H(Y) = Y \oplus F \).
• Because it’s isotropic \( \psi \) is a character of \( H(Y) \), acting on the second component.
• So we can induce the character \( \psi \) to a representation on \( H(W) \) and we a representation \( (\rho, S_Y) \).
• Now suppose that \( X + Y = H \) and both \( X \) and \( Y \) are isotropic.
• This is called a complete polarization of \( H \).
• It turns out that there’s an isomorphism \( S_Y \to S(X) \), where \( S(X) \) are compactly supported, locally constant functions on \( X \); a lovely space!
• If you trace through the isomorphism you can write down how \( H(W) \) acts on this space, its:

\[
\rho(h)f(x_0) = \rho(x + y, t)f(x_0) = \psi(t + <x_0, y> + \frac{1}{2} <x, y>)f(x_0 + x)
\]

• So this is the model \( (\rho, S(X)) \) of representations of the Heisenberg group.

We’re going to use this to create models of the Weil representation, which are representations of \( Sp(W) \) or, more precisely \( Mp(W) \)

We come to our first model, the Schrodinger model.

**The Schrodinger Model**

The Schrodinger model is a representation \( (\omega, S(X)) \) of \( Sp(W) \). Let \( g \in Sp(W) \). Then \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( a \in GL(X), d \in GL(Y), c \in Hom(X, Y), d \in Hom(Y, X) \).

Let \( f(x) \in S(X) \). Then we can define some the action

\[
\omega(g)f(x) = \int_{Y/ker(c)} \psi\left( \frac{1}{2} <ax, bx> - <bx, cy> + \frac{1}{2} <cy, dy> \right) \cdot f(ax+cy) d\mu(y)
\]

where \( \mu \) preserves the \( L^2 \) norm. Kudla calls this operation \( r(g) \)
But much better formulas for this are given by
\[
\omega\left(\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}\right)f(x) = |\det A|^{\frac{1}{2}} f(tAx)
\]
\[
\omega\left(\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}\right)f(x) = \psi\left(\begin{pmatrix} X & B \end{pmatrix}f(x)
\right)
\]
\[
\omega\left(\begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix}\right)f(x) = \gamma f(x)
\]
\[
\omega\left(\begin{pmatrix} -I & B \\ 0 & -I \end{pmatrix}\right)f(x) = \int_X \psi(<x',x>)f(b^{-1}x')dx'
\]

These define a projective representation. You can check that for any \(h \in H(W)\),
\[
\omega(g)\rho(h)\omega(g)^{-1} = \rho(h^g)
\]

another way to say this is that \((g,\omega(g)) \in \widetilde{Sp}(W)\).

**The Mixte Schrodinger Model**

Skipte

- This one is sort of complicated with lots of references to Chapter I which I didn’t print. The point of it is, that its something simple when you restrict to a certain maximal parabolic subgroup, in case that’s what you’re interested in.

**The Lattice Model**

- The Lattice Model is used to prove Howe duality
- Suppose now that \(F\) doesn’t have residue characteristic 2.
- Let \(A \subset W\) be a self-dual lattice \(A = A^\perp\) meaning that \(A = \{x \in W | \psi(<x,a>) = 1 \forall a \in A\}\)
- we create the character \(\psi_A\) by the projection of \(\psi\), that is \(\psi_A(w,t) = \psi(t)\).

\[
\psi_A = 1 \text{ on } (A,0) \subset H(W) \text{ since }
\]
\[
\psi_A((a_1,t_1)(a_2,t_2)) = \psi_A(t_1 + t_2)
\]

and
\[
\psi((a_1,t_1)(a_2,t_2)) = \psi(t_1 + t_2 + \frac{1}{2} < a_1,a_2 >)
\]

so if \(\frac{1}{2}\) is a unit, \(\psi_A\) is actually a character. \(\frac{1}{2}\) is a unit when the residue characteristic isn’t 2. (so, why do we use the factor \(\frac{1}{2}\)?)

- The lattice model is basically compact induction of this character \(\psi_A\) to \(H(W)\)
- Consider the space of functions \(S_A \subset S(W)\) , locally constant and compactly supported functions such that

\[
f(a + w) = \psi\left(\frac{1}{2} < w,a >\right)f(w)
\]

then we can define a representation of \(Sp(W)\) on this space by

\[
\omega(g)f(w,t) = \sum_{a \in A/gA \cap A} \psi(<a,w>)f(g^{-1}(a + w,t))
\]

- This representation has central character \(\psi\) so you just need to show that it’s irreducible. There’s some trick to show this in Takeda’s paper. And
that’s how you show that this is actually a model for the weil representation. You can do the direct check that \( \omega(g)\rho(h)\omega(g)^{-1} = \rho(h^g) \) as before. It’s a huge mess.

• Again, the condition:

\[
\omega(g)\rho(h)\omega(g)^{-1} = \rho(h^g).
\]

• The nice thing about this model is that the stabilizer of \(A, K\) is a maximal compact subgroup of \(Sp(W)\) and it behaves simply as

\[
\omega(g)f(w) = f(g^{-1}w)
\]

• **Relationship to supercuspidal representations**

• In Bump, section 4.8. We can take any finite dimensional vector space over a local field \(F\) with a symmetric inner product.

• Let \( W = V \times V \) with alternating form \( B(v_1, v_2) - B(v_2, v_1) \) and the Heisenberg group \(H(W)\) as before.

• Then the set \( X = \{(v, v)\} \subset W\) is a maximal isotropic space and there’s an action of \(Sl(2, F)\) on \(V \times V\) by \(g.(v_1, v_2, t) = (av_1 + bv_2, cv_1 + dv_2, t)\)

• as before, we can form the projective weil representation using the schrodinger model and show that it satisfies that thing.

• But in this case, the metaplectic group that makes this a true representation is split if \(V\) is even dimensional.

• In other words, the cocyle class is trivial.

• If we consider \(Sl(2, F) \subset Sp(W)\) as the set of block diagonal matrices, and if \(W = V \times V\), and \(V\) is even dimensional then the cocyle attached to \(Sp(W)\) is trivial when restricted to this subgroup.

• This gives us lots of supercuspidal representations of \(Sl(2, F)\).

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[| Prasad’s Notes

[|Kudla’s Notes

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