1. Reductive Dual Pairs

**Definition 1.** A reductive dual pair in $\text{Sp}(W)$ is a pair of subgroups $G, G' \subset \text{Sp}(W)$ such that $G, G'$ are reductive, $Z_{\text{Sp}(W)}(G) = G'$ and vice versa.

**Definition 2.** $(G, G')$ is reducible if there exist an orthogonal decomposition of $W = W_1 \oplus W_2$ and dual reductive pairs $(G_1, G'_1) \subset \text{Sp}(W_1)$, $(G_2, G'_2) \subset \text{Sp}(W_2)$ such that $(G, G') = (G_1 \times G_2, G'_1 \times G'_2)$.

We want to classify the irreducible reductive pairs.

**Example 3.** $(G = \text{Sp}(W), G' = Z(\text{Sp} W) = \{\pm 1\})$ is a dual reductive pair. This is an easy example but it matters.

**Example 4.** $(\text{Sp}(W), O(V)) \subset \text{Sp}(V \otimes W)$.

2. Type I

Let us construct Type I dual pairs, which generalize the last example. We start with a division algebra $D$ over $F$, with an involution $\sigma$. There are three cases:

- $D = F$, $\sigma = \text{Id}$;
- $D = E$ a quadratic extension over $F$, $\sigma =$ conjugation;
- $D = B$ a quaternion algebra over $F$, $\sigma =$ conjugation.

Let $W$ be a finite-dimensional left $D$-vector space. We choose $\epsilon = \pm 1$, and $\langle \cdot, \cdot \rangle$ an $\epsilon$-skew Hermitian form, i.e.

\[
\langle ax, by \rangle = a \langle x, y \rangle b^\sigma, \quad \langle x, y \rangle = -\epsilon \langle y, x \rangle^\sigma.
\]

Let $V$ be a right $D$-vector space and $w$ be an $\epsilon$-Hermitian form. Then

\[
\langle xa, yb \rangle = a^\sigma \langle x, y \rangle b, \quad \langle x, y \rangle = \epsilon \langle y, x \rangle^\sigma.
\]

Let $\mathbb{W} = V \otimes_D W$ be equipped with the symplectic form

\[
\langle \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle \rangle = \text{tr}_{D/F}(\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle).
\]

This makes $\mathbb{W}$ into a symplectic space, and $(U(W), U(V))$ is a dual pair in $\text{Sp}(\mathbb{W})$.

In the classification, this corresponds to $GG'$ acting on $W$ irreducibly.

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3. Type II

Let $D$ be a division algebra over $F$, but we don’t need an involution this time. Let $A,B$ be finite-dimensional right vector spaces. Then we define

$$A^* := \text{Hom}_D(A,D)$$

which is a left $D$-vector space, and similarly for $B^*$. Let $X = A \otimes B^*$ and $Y = B \otimes A^*$. Then we have a nondegenerate pairing $X \times Y \to F$

$$(a \otimes b^*, b \otimes a^*) = \text{tr}_{D/F}((a^*, a), (b^*, b))$$

where $(a^*, a)$ and $(b^*, b)$ are just the canonical pairings with dual spaces.

Then $W = X \oplus Y$ is a symplectic space with

$$\langle \langle x_1 + y_1, x_2 + y_2 \rangle \rangle = (x_1, y_2) - (x_2, y_1),$$

and $(\text{GL}_D(A), \text{GL}_D(B))$ is a reductive pair.

In the classification, this corresponds to $G \times G'$ acting on $W$ reducibly.

**Theorem 5.** All irreducible dual pairs are Type I or II.

We will not prove this theorem.

4. Local Theta Correspondence

Now we have the set-up to talk about the local theta correspondence. The key lemma is:

**Lemma 6.** Let $\tilde{G}, \tilde{G}'$ be the preimages of $G, G'$ under $\tilde{\text{Sp}}(W) \to \text{Sp}(W)$. Then $\tilde{G}$ and $\tilde{G}'$ commute. In particular, we have a surjection $\tilde{G} \times \tilde{G}' \to G \times G'$.

Thus we can pull back any representation $\pi_{\tilde{G}G'}$ of $\tilde{G}G'$ and obtain $\pi \otimes \pi'$, where $\pi$ is a representation of $\tilde{G}$ and $\pi'$ is a representation of $\tilde{G}'$.

**Definition 7.** Let $H \subset \text{Sp}(W)$. We define

$$R_\psi(\tilde{H}) := \{ \text{smooth irreducible representations } \pi \text{ of } \tilde{H} | \text{Hom}_{\tilde{H}}(S, \pi) \neq 0 \}$$

where $S$ is the Weil representation (dependent on $\psi$).

From the above, we have a map $R_\psi(G \times G') \to R_\psi(\tilde{G}) \times R_\psi(\tilde{G}')$.

The following theorem was conjectured by Howe and proved by Waldspurger.

**Theorem 8.** The image of $R_\psi(G \times G') \to R_\psi(\tilde{G}) \times R_\psi(\tilde{G}')$ is the graph of a bijection $R_\psi(\tilde{G}) \to R_\psi(\tilde{G}')$.

There are better things to say this. For a more concrete construction, let $\pi$ be an irreducible representation of $\tilde{G}$.

**Definition 9.** We define

$$N(\pi) = \bigcap_{\lambda \in \text{Hom}_{\tilde{G}}(S, \pi)} \ker \lambda$$

and

$$S(\pi) = S/N(\pi)$$

to be the maximal quotient where $\tilde{G}$ acts as a multiple of $\pi$. 

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Since $\widetilde{G}'$ preserves each $\lambda$, it acts on $S(\pi)$ and we can write
\[ S(\pi) = \pi \otimes \Theta_\psi(\pi) \]
as representations of $\widetilde{G}'$.

**Theorem 10** (Howe duality). (i) Either $\Theta_\psi(\pi) = 0$ or $\Theta_\psi(\pi)$ is admissible of finite length.
(ii) If $\Theta_\psi(\pi) \neq 0$, then it has a unique irreducible quotient, i.e. there exists unique $\Theta^*_\psi(\pi)$ such that
\[ \Theta_\psi(\pi)/\Theta^*_\psi(\pi) =: \theta_\psi(\pi) \]
is irreducible.
(iii) If $\theta_\psi(\pi_1) = \theta_\psi(\pi_2)$, then $\pi_1 = \pi_2$.

5. Small Examples

We will look at some small examples of the form $(O(V), \text{Sp}(W))$, where $\dim W = 2$. Thus $\text{Sp}(W) = \text{SL}_2$.

First we look at the case $\dim V = 1$, i.e. $V = F$ with $q(x) = ax^2$. Then $O(V) = \{\pm 1\}$, which has two irreducible representations. Let $\omega_\pi$ be the action of $\text{Sp}(W)$ on the Schwarz functions $S(F)$. Then $\omega_\psi = S_{\text{even}} \oplus S_{\text{odd}}$ (into even and odd functions) as irreducible representations of $\text{Sp}(W) = \text{SL}_2$.

Let’s go up one more dimension: $\dim V = 2$. We take $V = E$ to be a quadratic extension of $F$, with quadratic form $N_{E/F}$. Then $O(V)$ is the semidirect product of $E^1 = \{e \in E | N(e) = 1\}$ and $\text{Gal}(E/F)$, and so we can take characters of $E^1$ and use them to generate representations of $O(V)$. Let $\rho : E^1 \to \mathbb{C}^\times$.

(a) $\rho^2 \neq 1$. Then $\text{Ind}_{E^1}^{O(V)} \rho$ is irreducible and
\[ \text{Ind}_{E^1}^{O(V)} \rho \cong \text{Ind}_{E^1}^{O(V)} \rho^{-1}. \]

(b) $\rho^2 = 1$. Then this gives two different characters of $O(V)$, corresponding to the plus and minus eigenspaces under the Galois element.

Let’s look at the action of $\text{SL}_2$ on the Schwarz functions $S(E)$. We define
\[ S_\rho(E) = \{ f \in S(E) | f(ge) = \rho(g)f(e), g \in E^1 \}. \]
I claim that we can decompose the action in terms of direct sums of these. Indeed, we have a projection $P_\rho : S(E) \to S_\rho(E)$ given by
\[ P_\rho(f)(e) = \int_{E^1} \rho(g)f(ge)dg. \]
When $\rho^2 \neq 1$, these will all be irreducible.