**SOLUTIONS TO HW5**

Stein and Shakarchi, Chapter 3

**Problem 1.** (a) Consider the following sequence of functions: \( f_n(z) = \frac{z}{n} \). We then have \( f_n(0) = 0 \) for all \( n \). Now let \( r > 0 \) be a positive real number, and choose \( n \) such that \( \frac{1}{n} < r \), so that \( D_r(0) \supset D_{1/n}(0) \). We claim that the set \( f_n(\mathbb{D}) \) cannot contain \( D_r(0) \). To see this, assume that \( D_r(0) \subset f_n(\mathbb{D}) \), and choose \( w_0 \in D_r(0) \setminus D_{1/n}(0) \). Since \( w_0 \) is in \( f(\mathbb{D}) \), there exists \( z_0 \in \mathbb{D} \) such that \( f_n(z_0) = \frac{z_0}{n} = w_0 \). Therefore, by choice of \( w_0 \), we have

\[
1 \leq |w_0| = \left| \frac{z_0}{n} \right| < \frac{1}{n},
\]

a contradiction. Thus, the assumption \( f(0) = 0 \) alone is too weak to guarantee that \( f(\mathbb{D}) \) contains a fixed open disc. Note that \( f_n'(0) = \frac{1}{n} \), which goes to 0 as \( n \) approaches infinity.

(b) Consider now the function \( f_\varepsilon(z) = \varepsilon(e^{z/\varepsilon} - 1) \), where \( \varepsilon > 0 \). We have \( f_\varepsilon(0) = 0 \), and \( f_\varepsilon'(0) = 1 \). Let \( r > 0 \) be a positive number, and choose \( 0 < \varepsilon < r \), so that \( D_r(0) \supset D_\varepsilon(0) \). We claim that \( f_\varepsilon(\mathbb{D}) \) cannot contain \( D_r(0) \). To prove this, assume \( D_r(0) \subset f_\varepsilon(\mathbb{D}) \), so that \( -\varepsilon = f_\varepsilon(z_0) \) for some \( z_0 \in \mathbb{D} \). We then have

\[
\varepsilon(e^{z_0/\varepsilon} - 1) = f_\varepsilon(z_0) = -\varepsilon \iff e^{z_0/\varepsilon} - 1 = -1 \iff e^{z_0/\varepsilon} = 0,
\]

which is a contradiction. Therefore, even the additional condition \( f'(0) = 1 \) is too weak to guarantee that \( f(\mathbb{D}) \) contains a fixed open disc.

(c) Let \( h(z) = \frac{1}{z} + c_0 + c_1 z + \ldots = \sum_{n=-1}^\infty c_n z^n \), where we set \( c_{-1} = 1 \), and assume \( h(z) \) is analytic and injective on \( \mathbb{D} \setminus \{0\} \). Let \( D_\rho(0) \) be a small disc of radius \( \rho \), and consider the image \( h(D_\rho(0) \setminus \{0\}) \). The boundary of \( h(D_\rho(0) \setminus \{0\}) \) is given by \( h(\rho e^{i\theta}), \theta \in [0, 2\pi] \). Note that since \( h(z) \) is injective, \( h(\rho e^{i\theta}) \) is a Jordan curve. We compute its orientation. Fix \( \zeta \notin h(D_\rho(0) \setminus \{0\}) \), and consider the integral

\[
\frac{1}{2\pi i} \int_{h(\rho e^{i\theta})} \frac{dz}{z - \zeta}.
\]

This integral computes the number of times \( h(\rho e^{i\theta}) \) winds around the point \( \zeta \) (i.e., this is the winding number). Making the change of variables \( z = h(z') \) and using the argument principle, we obtain

\[
\frac{1}{2\pi i} \int_{h(\rho e^{i\theta})} \frac{dz}{z - \zeta} = \frac{1}{2\pi i} \int_{|z'|=\rho} \frac{h'(z')}{h(z') - \zeta} \, dz'
\]

\[
= \frac{1}{2\pi i} \int_{|z'|=\rho} \frac{(h(z') - \zeta')'}{h(z') - \zeta} \, dz'
\]

\[
= \#\{\text{zeros of } h(z) - \zeta \text{ in } |z| = \rho\} - \#\{\text{poles of } h(z) - \zeta \text{ in } |z| = \rho\}
\]

\[= -1.\]

Therefore, \( h(\rho e^{i\theta}) \) has a negative orientation.
Now, the area of \( h(D_\rho(0) \setminus \{0\})^\C \) is finite, and to compute this we use Green’s theorem:

\[
\text{Area}(h(D_\rho(0) \setminus \{0\})^\C) = \int_{h(D_\rho(0) \setminus \{0\})^\C} dA = -\frac{1}{2} \int_{h(\rho e^{i\theta})} -y \, dx + x \, dy = -\frac{1}{2} \Im \left( \int_{h(\rho e^{i\theta})} \overline{z} \, dz \right).
\]

Making the change of variables \( z = h(\rho e^{i\theta}) \), the latter integral becomes

\[
\int_{h(\rho e^{i\theta})} \overline{z} \, dz = i \int_{0}^{2\pi} h'(\rho e^{i\theta})\rho e^{i\theta} \, d\theta
\]

\[
= i \int_{0}^{2\pi} \left( \sum_{n \geq -1} c_n \rho^n e^{-in\theta} \right) \left( \sum_{m \geq -1} m c_m \rho^{m-1} e^{i(m-1)\theta} \right) \rho e^{i\theta} \, d\theta
\]

\[
= i \sum_{n, m \geq -1} m c_n c_m \rho^{n+m} \int_{0}^{2\pi} e^{i(m-n)\theta} \, d\theta
\]

\[
= 2\pi i \sum_{n \geq -1} n |c_n|^2 \rho^{2n}
\]

\[
= 2\pi i \left( -\frac{1}{\rho^2} + \sum_{n \geq 0} n |c_n|^2 \rho^{2n} \right).
\]

The equality (\( \ast \)) follows from the following computation:

\[
\int_{0}^{2\pi} e^{ik\theta} \, d\theta = \begin{cases} 0 & \text{if } k \neq 0, \\ 2\pi & \text{if } k = 0. \end{cases}
\]

Therefore, combining everything, we get

\[
0 \leq \text{Area}(h(D_\rho(0) \setminus \{0\})^\C) = -\frac{1}{2} \Im \left( \int_{h(\rho e^{i\theta})} \overline{z} \, dz \right)
\]

\[
= -\pi \left( -\frac{1}{\rho^2} + \sum_{n \geq 0} n |c_n|^2 \rho^{2n} \right),
\]

which implies

\[
\sum_{n \geq 0} n |c_n|^2 \rho^{2n} \leq \frac{1}{\rho^2}.
\]

Letting \( \rho \) tend to 1 gives the result.

(d) Assume that \( f(z) = z + a_2 z^2 + \ldots = \sum_{n \geq 1} a_n z^n \) (with \( a_1 = 1 \)) is injective, and consider the function \( \alpha(z) = f(z)/z \). The power series expansion of \( \alpha(z) \) shows that \( \lim_{x \to 0} \alpha(z) = 1 \). If there existed some \( z_0 \in \mathbb{D} \setminus \{0\} \) for which \( \alpha(z_0) = 0 \), then we would have \( f(z_0) = 0 \), which contradicts the injectivity of \( f(z) \). Therefore, \( \alpha(z) \) is nonvanishing on \( \mathbb{D} \), and we may define the square root \( \psi(z) \) by

\[
\psi(z) = e^{\frac{1}{2} \log \left( \frac{f(z)}{z} \right)}.
\]

Note that this function is well defined, and satisfies \( \psi(z)^2 = f(z)/z \) and \( \psi(0) = 1 \).
We now define 

\[ g(z) = z\psi(z^2). \]

Note that \( g(z) \) is odd by construction, and satisfies \( g(0) = 0 \) and 

\[ g'(0) = (\psi(z^2) + 2z^2\psi'(z^2))|_{z=0} = 1. \]

Moreover, we have \( g(z)^2 = z^2\psi(z^2)^2 = f(z^2) \). It remains to show that \( g(z) \) is injective. Assume there exist \( z_1, z_2 \in \mathbb{D} \) such that \( g(z_1) = g(z_2) \). We then have 

\[ f(z_1^2) = g(z_1)^2 = g(z_2)^2 = f(z_2^2), \]

which implies \( z_1^2 = z_2^2 \) by injectivity of \( f(z) \). Therefore, we must have \( z_2 = z_1 \) or \( z_2 = -z_1 \). If we are in the first case, we are done. Assume we are in the second case. Then we have 

\[ z_1\psi(z_1^2) = g(z_1) = g(z_2) = g(-z_1) = -z_1\psi(z_1^2), \]

and hence 

\[ 2z_1\psi(z_1^2) = 0. \]

Therefore, either \( z_1 = 0 \), and we are done, or \( \psi(z_1^2) = 0 \), which is a contradiction. This implies that \( g(z) \) is injective.

(e) We compute the first terms in a power series expansion of \( g(z)^{-1} \) (with notation as in the previous part). We have 

\[ \frac{f(z^2)}{z^2} = 1 + a_2z^2 + a_3z^4 + a_4z^6 + \ldots \]

\[ -\frac{1}{2} \log \left( \frac{f(z^2)}{z^2} \right) = -\frac{1}{2} \log(1 + (a_2z^2 + a_3z^4 + a_4z^6 + \ldots)) \]

\[ = -\frac{1}{2} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} (a_2z^2 + a_3z^4 + a_4z^6 + \ldots)^n \]

\[ = -\frac{1}{2} \left( a_2z^2 + a_3z^4 + a_4z^6 + \ldots - \frac{1}{2}(a_2z^2 + a_3z^4 + a_4z^6 + \ldots)^2 + \ldots \right) \]

\[ = \left( -\frac{a_2}{2} \right) z^2 + \left( -\frac{a_3}{2} + \frac{a_2^2}{4} \right) z^4 + \ldots \]

\[ g(z)^{-1} = z^{-1}\psi(z^2)^{-1} \]

\[ = z^{-1} e^{-\frac{1}{2} \log \left( \frac{f(z^2)}{z^2} \right)} \]

\[ = z^{-1} \left( \sum_{m \geq 0} \frac{1}{m!} \left( \left( -\frac{a_2}{2} \right) z^2 + \left( -\frac{a_3}{2} + \frac{a_2^2}{4} \right) z^4 + \ldots \right)^m \right) \]

\[ = z^{-1} \left( 1 + \left( -\frac{a_2}{2} \right) z^2 + \left( -\frac{a_3}{2} + \frac{a_2^2}{4} \right) z^4 + \ldots \right) \]

\[ + \frac{1}{2} \left( -\frac{a_2}{2} \right) z^2 + \left( -\frac{a_3}{2} + \frac{a_2^2}{4} \right) z^4 + \ldots \right)^2 + \ldots \]

\[ = z^{-1} + \left( -\frac{a_2}{2} \right) z + \left( -\frac{a_3}{2} + \frac{3a_2^2}{8} \right) z^3 + \ldots \]
Now, write \(1/g(z) = 1/z + c_0 + c_1z + \ldots\) as in part (c), with the coefficients \(c_n\) given by the expansion above. Since \(1/g(z)\) is analytic and injective on \(\mathbb{D} \setminus \{0\}\), we may apply the result of part (c) to get

\[
\sum_{n \geq 1} n |c_n|^2 \leq 1.
\]

Assume \(|a_2| > 2\). Since \(c_1 = -\frac{a_2}{2}\), we obtain

\[
1 < \left| -\frac{a_2}{2} \right|^2 = |c_1|^2 \leq |c_1|^2 + \sum_{n \geq 2} n |c_n|^2 \leq 1,
\]

a contradiction. Therefore, we must have \(|a_2| \leq 2\).

Now assume that \(|a_2| = 2\), so that \(a_2 = 2e^{i\varphi}\) for some \(\varphi \in [0, 2\pi]\). The computation above shows that we must have

\[
\sum_{n \geq 2} n |c_n|^2 = 0,
\]

so that \(c_n = 0\) for \(n \geq 2\). Hence, we have

\[
\frac{1}{g(z)} = \frac{1}{z - e^{i\varphi}z} \iff g(z) = \frac{z}{1 - e^{i\varphi}z^2}
\]

\[
\iff f(z^2) = g(z)^2 = \frac{z^2}{(1 - e^{i\varphi}z^2)^2}
\]

\[
\iff f(z) = \frac{z}{(1 - e^{i\varphi}z)^2}.
\]

On the other hand, if \(f(z) = \frac{z}{(1 - e^{i\varphi}z)^2}\), we can easily check that \(f(0) = 0, f'(0) = 1\), and that \(f(z)\) is injective. We may expand using a geometric series (which is valid since \(z \in \mathbb{D}\)):

\[
f(z) = \frac{z}{(1 - e^{i\varphi}z)^2} = \frac{z}{1 - 2e^{i\varphi}z + e^{2i\varphi}z^2 - \ldots} = 1 + 2e^{i\varphi}z + 3e^{2i\varphi}z^2 + \ldots.
\]

This shows that \(|a_2| = 2\).

(f) As before, we let \(h(z) = \sum_{n \geq -1} c_n z^n\) (with \(c_{-1} = 1\)), assume \(h(z)\) is analytic and injective on \(\mathbb{D} \setminus \{0\}\), and assume that \(z_j, \text{ for } j = 1, 2\) are two points such that \(z_j \notin h(\mathbb{D} \setminus \{0\})\).

We first state a general fact. Let \(A(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \ldots\) be a power series. We wish to compute (the first few terms of) \(A(z)^{-1}\). One way to do this is to write \(A(z)^{-1} = \beta_0 + \beta_1 z + \beta_2 z^2 + \ldots\) and take the product. This gives

\[
1 = A(z)A(z)^{-1} = (1 + \alpha_1 z + \alpha_2 z^2 + \ldots)(\beta_0 + \beta_1 z + \beta_2 z^2 + \ldots) = \beta_0 + (\beta_1 + \alpha_1 \beta_0)z + (\beta_2 + \alpha_1 \beta_1 + \alpha_2 \beta_0)z^2 + \ldots
\]
Note that this is just the Cauchy product. This shows, for example, that $\beta_0 = 1, \beta_1 = -\alpha_1, \beta_2 = -\alpha_2 + \alpha_1^2, \ldots$. Therefore, we have

$$A(z)^{-1} = 1 - \alpha_1 z + \ldots$$

There are many other ways to compute these coefficients: we can also compute the derivatives at 0 of $A(z)^{-1}$ using the inverse function theorem.

Now, let $z_j$ be one of the points not in the image of $h(z)$. Since $h(z)$ is injective, so is $h(z) - z_j$, and hence so is $1/(h(z) - z_j)$. Now consider the power series expansion of this function:

$$h(z) - z_j = \frac{1}{z} + (c_0 - z_j) + c_1 z + \ldots$$

$$= \frac{1}{z} (1 + (c_0 - z_j)z + c_1 z^2 + \ldots)$$

$$= z (1 + (c_0 - z_j)z + c_1 z^2 + \ldots)^{-1}$$

$$= z (1 + (-c_0 + z_j)z + \ldots).$$

In particular, we see that the function $1/(h(z) - z_j)$ takes the value 0 at $z = 0$, has derivative 1 at 0, and is injective. Applying part (e), we obtain $|z_j - c_0| \leq 2$, and therefore $|z_1 - z_2| \leq |z_1 - c_0| + |c_0 - z_2| \leq 4$ by the triangle inequality.

(g) Assume now that $f(z) = z + a_2 z^2 + \ldots$ is a function such that $f(0) = 0, f'(0) = 1$, and such that $f(z)$ is injective. Assume $w$ is not in the image of $f(z)$. Then the points $1/w$ and 0 are not in the image of the function $1/f(z)$, and therefore by part (f), we must have

$$\left| \frac{1}{w} - 0 \right| \leq 4 \iff |w| \geq \frac{1}{4}.$$ 

This shows that $f(\mathbb{D})$ must contain the open disc of radius $\frac{1}{4}$.

**Problem 2.** Let $u(x, y) = u(z)$ be a harmonic function on $\mathbb{D}$ which extends continuously to $\mathbb{D}$. Let $z_0 \in \mathbb{D}$. Let $T(z)$ denote the function

$$T(z) = \frac{z_0 - z}{1 - \overline{z_0} z}.$$ 

This is an example of a Möbius transformation. Since $|z_0| < 1$, $T(z)$ is holomorphic on $\mathbb{D}$. It will be convenient to know the inverse transformation $T^{-1}(z)$; to find it, we set $w = T(z)$ and solve for $z$:

$$w = \frac{z_0 - z}{1 - \overline{z_0} z} \iff w - w\overline{z_0} z = z_0 - z \iff z - w\overline{z_0} z = z_0 - w \iff z = \frac{z_0 - w}{1 - \overline{z_0} w}.$$ 

Therefore, if $w = T(z)$, then $z = T(w)$, or in other words, $T \circ T(z) = z$.

Now, since $u(z)$ is harmonic, there exists a holomorphic function $f(z)$ such that $\Re(f(z)) = u(z)$. To see this, all we have to do is find a function $v(z)$ which satisfies the following family
of partial differential equations:
\[
\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \\
\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}
\]

The existence of such a \(v(z)\) is equivalent to the vector field \((x, y) \mapsto -\begin{pmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{pmatrix}\) being the gradient of a function, which is in turn equivalent to \(u(z)\) being harmonic.

The upshot of this is that \(f(z) = u(z) + iv(z)\) is a holomorphic function, and therefore \(f(T(z))\) is holomorphic as well. This immediately implies that \(\Re(f(T(z))) = u(T(z)) =: u_0(z)\) is harmonic. Note that we may also compute \(\Delta(u(T(z)))\) directly to verify harmonicity.

We now apply the mean value theorem to \(u_0(z)\):
\[
u(z_0) = u_0(0) = \frac{1}{2\pi} \int_0^{2\pi} u_0(e^{i\theta}) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} u(T(e^{i\theta})) \, d\theta.
\]

We define a change of variables by \(e^{i\theta'} = T(e^{i\theta})\), which implies \(e^{i\theta} = T(e^{i\theta'}) = \frac{z_0 - e^{i\theta'}}{1 - \overline{z_0} e^{i\theta'}}\) by the computation above. Therefore,
\[
ie^{i\theta} \, d\theta = \frac{(1 - \overline{z_0} e^{i\theta'})(-ie^{i\theta'}) - (z_0 - e^{i\theta})(-i\overline{z_0} e^{i\theta'})}{(1 - \overline{z_0} e^{i\theta'})^2} \, d\theta' = \frac{-i(1 - |z_0|^2) e^{i\theta'}}{(1 - \overline{z_0} e^{i\theta'})^2} \, d\theta'
\]

\(\iff d\theta = \frac{1}{ie^{i\theta'}} - i(1 - |z_0|^2) e^{i\theta'} \, d\theta' = \frac{1}{z_0 - e^{i\theta'}} - (1 - |z_0|^2) e^{i\theta'} \, d\theta' = \frac{1 - |z_0|^2}{|z_0 - e^{i\theta'}|^2} \, d\theta'.\)

Substituting this into the integral gives
\[
u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(T(e^{i\theta})) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z_0|^2}{|z_0 - e^{i\theta'}|^2} u(e^{i\theta'}) \, d\theta',
\]
which gives the result.

Now let \(z_0 = re^{i\varphi}\). Then
\[
\frac{1 - |z_0|^2}{|e^{i\theta} - z_0|^2} = \frac{1 - r^2}{(e^{i\theta} - re^{i\varphi})(e^{-i\theta} - re^{-i\varphi})} = \frac{1 - r^2}{1 + r^2 - re^{i(\theta - \varphi)} - re^{i(\varphi - \theta)}} = \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \varphi)} = P_r(\theta - \varphi),
\]
where $P_r(\theta - \varphi)$ is the Poisson kernel. Therefore

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \varphi)} u(e^{i\theta}) \, d\theta = (u \ast P_r)(\varphi),$$

which shows that we can recover the values of the harmonic function $u(z)$ inside the unit disc by convolving $u$ with the Poisson kernel and evaluating on the boundary $\partial \mathbb{D}$. 
