SOLUTIONS TO HW4

Stein and Shakarchi, Chapter 3

Exercise 3. Let \( f(z) = \frac{e^{iz}}{z^2 + a^2} \), with \( a > 0 \), and let \( \gamma_R = \gamma_1 \cup \gamma_2 \), where \( \gamma_1 \) denotes the contour from \(-R\) to \( R\) along the real axis, and \( \gamma_2 \) is the semicircular contour \( Re^{it} \) for \( t \in [0, \pi] \). Take \( R > a \). On \( \gamma_2 \), we have (using the reverse triangle inequality)

\[
\left| \frac{e^{iz}}{z^2 + a^2} \right| \leq \frac{e^{-y}}{|z|^2 - a^2} \leq \frac{1}{R^2 - a^2}.
\]

Hence, we have

\[
\left| \int_{\gamma_2} \frac{e^{iz}}{z^2 + a^2} \, dz \right| \leq \sup_{z \in \gamma_2} \left( \left| \frac{e^{iz}}{z^2 + a^2} \right| \right) \cdot \text{length}(\gamma_2) \leq \frac{\pi R}{R^2 - a^2},
\]

which goes to zero as \( R \) goes to infinity. Therefore, using the Residue theorem and the fact that the only pole of \( f(z) \) inside \( \gamma_R \) is at \( ia \) (and is simple), we get

\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} \, dx = \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{iz}}{z^2 + a^2} \, dz
\]

\[
= 2\pi i \ \text{res}_{z=ia} \left( \frac{e^{iz}}{z^2 + a^2} \right)
\]

\[
= 2\pi i \lim_{z \to ia} (z - ia) \frac{e^{iz}}{z^2 + a^2}
\]

\[
= 2\pi i \frac{e^{ia}}{a + ia}
\]

\[
= \pi \frac{e^{-a}}{a}.
\]

Taking real parts of both sides gives

\[
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} \, dx = \frac{\pi e^{-a}}{a}.
\]

Exercise 8. Consider the integral

\[
\int_{0}^{2\pi} \frac{d\theta}{a + b \cos(\theta)}.
\]
with \( a, b \in \mathbb{R}, a > |b| \). By periodicity, the value of this integral does not change if we replace \( b \) with \(-b\), so we may assume \( b \geq 0 \). We set \( z = e^{i\theta} \); we then have
\[
\begin{align*}
\bar{z} &= e^{-i\theta} = z^{-1} \\
z \, dz &= ie^{i\theta} \, d\theta = iz \, d\theta \\
\cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}.
\end{align*}
\]

As \( \theta \) ranges over \([0, 2\pi]\), \( z \) will range over the unit circle. Therefore, making a change of variables, we get
\[
\int_0^{2\pi} \frac{d\theta}{a + b \cos(\theta)} = \int_{|z|=1} \frac{i^{-1}z^{-1}}{a + b \frac{z + z^{-1}}{2}} \, dz = -2i \int_{|z|=1} \frac{1}{bz^2 + 2az + b} \, dz.
\]

When \( b = 0 \), we arrive at the result using Cauchy’s integral formula. When \( b \neq 0 \), the roots of \( bz^2 + 2az + b \) occur at \( z = \frac{-a \pm \sqrt{a^2 - b^2}}{b} \). The only one of these roots occurring inside the unit circle is \( z = \frac{-a + \sqrt{a^2 - b^2}}{b} \), and this root is simple. Using the residue theorem gives
\[
\int_0^{2\pi} \frac{d\theta}{a + b \cos(\theta)} = -2i \int_{|z|=1} \frac{1}{bz^2 + 2az + b} \, dz = 4\pi \text{ res}_{z=-a+\sqrt{a^2-b^2}} \left( \frac{1}{bz^2 + 2az + b} \right)
\]
\[
= 4\pi \lim_{z \to -a+\sqrt{a^2-b^2}} \left( \frac{1}{bz^2 + 2az + b} \right) = \frac{1}{b \left( -a - \sqrt{a^2 - b^2} \right)} = \frac{2\pi}{\sqrt{a^2 - b^2}}.
\]

**Exercise 12.** Fix \( u \in \mathbb{C} \setminus \mathbb{Z} \), and let \( f(z) = \frac{\pi \cot(\pi z)}{(u+z)^2} \). Note that the poles of \( f(z) \) occur at \( z = -u \) (which is a double pole) and at all integers (which are simple poles). Fix \( N \geq |u| \), and consider the circle of radius \( N + \frac{1}{2} \). Note that since \( u \) is not an integer, we have \( N \geq 1 \).

We must now bound \( f(z) \) on \(|z| = N + \frac{1}{2}\). Let \( R \) denote the rectangle in the complex plane given by
\[
\frac{\sqrt{5}}{2} \leq x \leq 3 - \frac{\sqrt{5}}{2}, \\
-1 \leq y \leq 1.
\]

Note that \( R \) is compact and does not contain any integers. Since \( \cot(z) \) is \( \pi \)-periodic, we see that \( \cot(\pi z) \) takes the same values on all the rectangles \( n + R \), where \( n \in \mathbb{Z} \). Let \(|\cot(\pi z)| \leq M \) on \( R \). The rectangle \( R \) was constructed so that the intersection of the circle \(|z| = N + \frac{1}{2}\) and the strip \(-1 \leq y \leq 1\) is contained in the union of the rectangles \((N-1) + R\) and \((-N-2) + R\) (draw a picture of everything to see what is going on).
Consider the following sequence of inequalities:

\[
|\cot(\pi z)| = \left| \frac{\cos(\pi z)}{\sin(\pi z)} \right| = \left| \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} \right| = \left| \frac{e^{\Re(\pi iz)} + e^{\Re(-\pi iz)}}{e^{\Re(\pi iz)} - e^{\Re(-\pi iz)}} \right| \leq \left| \frac{e^{-\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}} \right|.
\]

When \( y \geq 1 \), then the last line is equal to \( \coth(\pi y) \), which is a decreasing function of \( y \). Therefore, when \( y \geq 1 \), we get the bound

\[
|\cot(\pi z)| \leq \coth(\pi y) \leq \coth(\pi).
\]

Likewise, when \( y \leq -1 \), the last line above is equal to \(-\coth(\pi y)\), which decreases as \( y \) goes to \(-\infty\). Therefore, when \( y \leq -1 \), we obtain

\[
|\cot(\pi z)| \leq -\coth(\pi y) = -\coth(-\pi) = \coth(\pi).
\]

Now let \( N \geq |u| \). Using the usual bounds, we get

\[
\left| \int_{|z|=N+\frac{1}{2}} \frac{\pi \cot(\pi z)}{(z + u)^2} \, dz \right| = \left| \int_{|z|=N+\frac{1}{2}, -1 \leq y \leq 1} \frac{\pi \cot(\pi z)}{(z + u)^2} \, dz \right| + \left| \int_{|z|=N+\frac{1}{2}, y \geq 1} \frac{\pi \cot(\pi z)}{(z + u)^2} \, dz \right| + \left| \int_{|z|=N+\frac{1}{2}, y \leq -1} \frac{\pi \cot(\pi z)}{(z + u)^2} \, dz \right|
\]

\[
\leq \pi \left( N + \frac{1}{2} \right) \sup_{|z|=N+\frac{1}{2}, -1 \leq y \leq 1} \left( \frac{\pi \cot(\pi z)}{(z + u)^2} \right) + \pi \left( N + \frac{1}{2} \right) \sup_{|z|=N+\frac{1}{2}, y \geq 1} \left( \frac{\pi \cot(\pi z)}{(z + u)^2} \right) + \pi \left( N + \frac{1}{2} \right) \sup_{|z|=N+\frac{1}{2}, y \leq -1} \left( \frac{\pi \cot(\pi z)}{(z + u)^2} \right)
\]

\[
\leq \pi \left( N + \frac{1}{2} \right) \frac{\pi M}{(N + \frac{1}{2} - |u|)^2} + \pi \left( N + \frac{1}{2} \right) \frac{\pi \coth(\pi)}{(N + \frac{1}{2} - |u|)^2} + \pi \left( N + \frac{1}{2} \right) \frac{\pi \coth(\pi)}{(N + \frac{1}{2} - |u|)^2}
\]

\[
= \pi^2 \left( N + \frac{1}{2} \right) \frac{M + 2 \coth(\pi)}{(N + \frac{1}{2} - |u|)^2}.
\]

Therefore, we see that the integral goes to zero as \( N \) approaches infinity.
On the other hand, we may compute the integral using the Residue theorem. The residues are given by

\[ \text{res}_{z=-u} \left( \frac{\pi \cot(\pi z)}{(z+u)^2} \right) = \lim_{z \to -u} \frac{d}{dz} \left( \pi \cot(\pi z) \right) \]

\[ = \frac{-\pi^2}{\sin(\pi u)^2} \]

\[ \text{res}_{z=n} \left( \frac{\pi \cot(\pi z)}{(z+u)^2} \right) = \lim_{z \to n} \frac{\pi \cot(\pi z)}{(z+n)(z+u)^2} \]

\[ = \frac{\pi}{(n+u)^2} \lim_{z \to n} \frac{z-n}{\tan(\pi z) - \tan(\pi n)} \]

\[ = \frac{\pi}{(n+u)^2} \left( \frac{d}{dz} \tan(\pi z) \bigg|_{z=n} \right)^{-1} \]

\[ = \frac{\pi}{(n+u)^2} \left( \cos(\pi n)^2 \right)^{-1} \]

\[ = \frac{1}{(n+u)^2}, \]

where in the last lines, \( n \in \mathbb{Z} \), and we have used the definition of the derivative. Combining everything, we obtain

\[ \frac{-\pi^2}{\sin(\pi u)^2} + \sum_{n=-N}^{N} \frac{1}{(n+u)^2} = \frac{1}{2\pi i} \int_{|z|=N+\frac{1}{2}} \frac{\pi \cot(\pi z)}{(z+u)^2} dz. \]

Letting \( N \) go to infinity gives us the desired result.

**Exercise 14.** Assume \( f(z) \) is entire and injective. We write

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \]

where \( a_n \in \mathbb{C} \). Since \( f \) is entire this series converges on all of \( \mathbb{C} \). Assume first that \( f(z) \) is not a polynomial, so that there are infinitely many coefficients such that \( a_n \neq 0 \). Then the function

\[ g(z) = f \left( \frac{1}{z} \right) = \sum_{n=0}^{\infty} a_n z^{-n}, \]

is holomorphic on \( \mathbb{C} \setminus \{0\} \), has an essential singularity at \( z = 0 \) (since infinitely many \( a_n \) are nonzero), and is injective as a map \( g : \mathbb{C} \setminus \{0\} \to \mathbb{C} \).

Now, consider set \( g(\{z \in \mathbb{C} : 0 < |z| < 1\}) \). Since \( g(z) \) has an essential singularity at 0, this set is dense in the complex plane. On the other hand, \( g(z) \) is holomorphic on \( \mathbb{C} \setminus \{0\} \), and therefore \( g(\{z \in \mathbb{C} : |z| > 1\}) \) will be open, by the open mapping theorem. By definition of density, these two sets must intersect: \( g(\{z \in \mathbb{C} : 0 < |z| < 1\}) \cap g(\{z \in \mathbb{C} : |z| > 1\}) \neq \emptyset \).

Since the two sets \( \{z \in \mathbb{C} : 0 < |z| < 1\}, \{z \in \mathbb{C} : |z| > 1\} \) are disjoint, this contradicts the injectivity of \( g(z) \).
Therefore, we may assume $f(z)$ is a polynomial of degree $n$. Assume $n \geq 2$. By the Fundamental Theorem of Algebra, $f(z)$ will have exactly $n$ roots $\alpha_1, \ldots, \alpha_n$ (possibly with repetition). If two of these roots are distinct, say $\alpha_1 \neq \alpha_2$, this means that the map $z \mapsto f(z)$ is not injective, since $f(\alpha_1) = f(\alpha_2) = 0$. Therefore, we must have $f(z) = a(z - \alpha_1)^n$. In this case $f(1 + \alpha_1) = a = f(e^{\frac{2\pi i}{n}} + \alpha_1)$, and again $f(z)$ is not injective. Therefore, $n$ must equal 1.

**Exercise 17.** Assume $f(z)$ is non-constant and holomorphic on a set containing the closed unit disc $D$.

(a) Assume that $|f(z)| = 1$ whenever $|z| = 1$, and let $w_0 \in D$. Applying Rouché’s theorem to the functions $f(z)$ and $g(z) = -w_0$, we see that

$$|f(z)| = 1 > |w_0| = |g(z)|$$

for all $z \in \partial D$, and therefore $f(z)$ and $f(z) + g(z) = f(z) - w_0$ have the same number of zeros in $D$. Hence, if $f(z) = 0$ has a solution, then $f(z) = w_0$ has a solution.

We are reduced to showing $f(z) = 0$ for some $z \in D$. Assume not. Then the function $1/f(z)$ is holomorphic on $D$, and the maximum modulus principle implies

$$1/\inf_{z \in D}(|f(z)|) = \sup_{z \in D}(1/|f(z)|) \leq \sup_{z \in \partial D}(1/|f(z)|) = 1.$$

Applying the maximum modulus principle to $f(z)$, we obtain

$$\sup_{z \in D}(|f(z)|) \leq \sup_{z \in \partial D}(|f(z)|) = 1.$$

Hence, if $z \in D$, we have

$$1 \leq \inf_{z \in D}(|f(z)|) \leq |f(z)| \leq \sup_{z \in D}(|f(z)|) \leq 1,$$

which implies $f(z)$ is constant by Chapter 1, Problem 13. We thus have a contradiction, and therefore the image of $f$ contains $D$.

(b) Assume that $|f(z)| \geq 1$ whenever $|z| = 1$ and that there exists $z_0 \in D$ such that $|f(z_0)| < 1$. As above, it suffices to show that $f(z) = 0$ has a solution. Assume it doesn’t. Since $1/|f(z)| \leq 1$ when $|z| = 1$, we proceed as above to obtain

$$1/\inf_{z \in D}(|f(z)|) = \sup_{z \in D}(1/|f(z)|) \leq \sup_{z \in \partial D}(1/|f(z)|) \leq 1.$$

This implies

$$1 \leq \inf_{z \in D}(|f(z)|) \leq |f(z_0)| < 1,$$

which gives a contradiction. Thus the image of $f$ must contain the unit disc.

**Exercise 22.** Assume $f(z)$ is holomorphic on $D$ and has a continuous extension to $\overline{D}$, such that $f(z) = \frac{1}{z}$ on $\partial D$. We will prove this problem in two ways.

First Solution. Consider the function $g(z) = f(z) - z$, and write $g(z) = u(x, y) + iv(x, y)$. Since the real and imaginary parts of holomorphic functions are harmonic, we see that $u$ must satisfy

$$\begin{cases}
\Delta u(x, y) = 0 & \text{on } D, \\
u(x, y)|_{\partial D} = 0 & \text{on } \partial D,
\end{cases}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator. By the maximum principle for harmonic functions, the maximum and minimum values of $u(x, y)$ are attained on the boundary $\partial D$. 

(see Strauss’s *Partial Differential Equations: An Introduction* for a reference). Since \( u(x,y) \) restricted to the boundary is 0, we see that \( u(x,y) = 0 \) on all of \( \overline{D} \). By Chapter 1, Problem 13, \( g(z) \) must be constant on \( D \). Hence, we have \( f(z) = z + \alpha \), for some (purely imaginary) \( \alpha \); letting \( z \) approach 1 shows

\[
1 + \alpha = \lim_{z \to 1} z + \alpha = \lim_{z \to 1} f(z) = f(1) = \frac{1}{i} = 1,
\]

so \( \alpha = 0 \). Letting \( z \) approach \( i \) shows that

\[
i = \lim_{z \to i} z = \lim_{z \to i} f(z) = f(i) = \frac{1}{i} = -i,
\]

and we obtain a contradiction. Hence, no such function can exist.

**Second Solution.** Let \( 0 < t < 1 \), and consider the function on \( D \) defined by \( f_t(z) = f(tz) \). Clearly \( f_t(z) \) is holomorphic. Since \( f(z) \) is continuous on \( \overline{D} \), there exists some \( M \) such that \( |f(z)| \leq M \) for all \( z \in \overline{D} \). As multiplication by \( t \) contracts \( D \), we have \( |f_t(z)| \leq M \) for all \( z \in \overline{D} \) and all \( 0 < t < 1 \). Obviously we have \( f_t(z) \to f(z) \) pointwise as \( t \to 1^- \). Therefore, by the Dominated Convergence Theorem (see Folland’s *Real Analysis*), we have

\[
\lim_{t \to 1^-} \int_{C(0,1)} f_t(z) \, dz = \int_{C(0,1)} f(z) \, dz,
\]

where \( C(0,1) \) denotes the circle of radius 1 centered at 0. Now, since \( f(z) \) is holomorphic on \( D \), we evaluate the right hand side using Cauchy’s integral theorem and a change of variables:

\[
\int_{C(0,1)} f_t(z) \, dz = \int_{C(0,1)} f(tz) \, dz
\]

\[
= \int_{C(0,1)} t^{-1} f(z') \, dz'
\]

\[
= 0.
\]

On the other hand, Cauchy’s integral formula gives us

\[
\int_{C(0,1)} f(z) \, dz = \int_{C(0,1)} \frac{1}{z} \, dz = 2\pi i.
\]

So we obtain

\[
0 = \lim_{t \to 1^-} \int_{C(0,1)} f_t(z) \, dz = \int_{C(0,1)} f(z) \, dz = 2\pi i,
\]

a contradiction.