SOLUTIONS TO HW2

Stein and Shakarchi, Chapter 1

Exercise 13. Let $f$ be a holomorphic function on an open set $\Omega$, and write $f(z) = u(x, y) + iv(x, y)$ as usual, with $u$ and $v$ differentiable.

(a) Assume $\Re(f) = u(x, y)$ is constant, so that
\[
\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0.
\]

Since $f$ is holomorphic, it satisfies the Cauchy-Riemann equations, and so we also get
\[
\frac{\partial v}{\partial y} = 0 \quad \text{and} \quad -\frac{\partial v}{\partial x} = 0,
\]
which implies that $v(x, y)$ is constant, and hence $f$ is constant.

(b) The proof is the same as the one in (a).

(c) Assume $|f|$ is constant, so that $|f(z)|^2 = u(x, y)^2 + v(x, y)^2$ is also constant. We differentiate this relation with respect to $x$ and $y$ and use the chain rule and the Cauchy-Riemann equations to get the following equations:
\[
\begin{align*}
2 \left( u(x, y) \frac{\partial u}{\partial x} - v(x, y) \frac{\partial u}{\partial y} \right) \overset{\text{CR eqs.}}{=} 2 \left( u(x, y) \frac{\partial u}{\partial x} + v(x, y) \frac{\partial v}{\partial x} \right) = 0 \\
2 \left( u(x, y) \frac{\partial u}{\partial y} + v(x, y) \frac{\partial u}{\partial x} \right) \overset{\text{CR eqs.}}{=} 2 \left( u(x, y) \frac{\partial u}{\partial y} + v(x, y) \frac{\partial v}{\partial y} \right) = 0
\end{align*}
\]

Multiplying the first equation by $\frac{u(x, y)}{2}$ and the second equation by $\frac{v(x, y)}{2}$ and adding, we obtain
\[
(u(x, y)^2 + v(x, y)^2) \frac{\partial u}{\partial x} = 0.
\]

If $u(x, y)^2 + v(x, y)^2 = 0$, then each of $u(x, y)$ and $v(x, y)$ are 0, so the claim follows. Otherwise, $u(x, y)^2 + v(x, y)^2 \neq 0$, and we see that $\frac{\partial u}{\partial x} = 0$. A similar argument shows that $\frac{\partial u}{\partial y} = 0$, so that $u(x, y)$ is constant, and we conclude using part (a).

Remark. Part (c) can be done more elegantly as follows. The two equations above are equivalent to the matrix equation
\[
\begin{pmatrix}
u(x, y) & -v(x, y) \\
v(x, y) & u(x, y)
\end{pmatrix}
\begin{pmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y}
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

The matrix on the left has determinant $u(x, y)^2 + v(x, y)^2$. Therefore, if $u(x, y)^2 + v(x, y)^2 = 0$, we proceed as before, and if $u(x, y)^2 + v(x, y)^2 \neq 0$, then we may invert the matrix to conclude that $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. 

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Exercise 14. Let \( \{a_n\}_{n=1}^{N} \) and \( \{b_n\}_{n=1}^{N} \) be two sequences of complex number, and set \( B_k = \sum_{n=1}^{k} b_n \). Note that \( B_n - B_{n-1} = b_n \). We then have (by reindexing the sum on the third line)

\[
\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n (B_n - B_{n-1})
\]

\[
= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1}
\]

\[
= a_N B_N - a_M B_{M-1} + \sum_{n=M}^{N-1} a_n B_n - \sum_{n=M+1}^{N} a_n B_{n-1}
\]

\[
= a_N B_N - a_M B_{M-1} + \sum_{n=M}^{N-1} (a_n - a_{n+1}) B_n.
\]

Exercise 15. Roughly, the idea is to apply the previous exercise with \( a_n = r^n \) and \( b_n = a_n \). Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence such that \( \sum_{n=1}^{\infty} a_n \) converges, and set \( A_k = \sum_{n=1}^{k} a_n \) for \( k \geq 1 \), \( A_0 = 0 \), and \( A = \lim_{n \to \infty} A_n = \sum_{n=1}^{\infty} a_n \). Then we have, for \( r < 1 \),

\[
\sum_{n=1}^{N} r^n a_n = r^N A_N - r A_0 - \sum_{n=1}^{N-1} (r^{n+1} - r^n) A_n
\]

\[
= r^N A_N + (1 - r) \sum_{n=1}^{N-1} r^n A_n
\]

\[
= r^N A_N + (1 - r) \sum_{n=0}^{N-1} r^n A_n,
\]

where the last line holds since \( A_0 = 0 \). Since we are interested in the limit as \( r \) approaches 1 from the left, we may assume that \( 0 < r < 1 \). Under this assumption, taking the limit as \( N \) goes to infinity in the equation above, we obtain

\[
\sum_{n=1}^{\infty} r^n a_n = (1 - r) \sum_{n=0}^{\infty} r^n A_n.
\]
Now let \( \varepsilon > 0 \) be given, and choose \( N_0 \) so that \( |A - A_n| \leq \frac{\varepsilon}{2} \) for \( n > N_0 \). Then we have

\[
\sum_{n=1}^{\infty} r^n a_n - A = (1 - r) \sum_{n=0}^{\infty} r^n A_n - A \\
\leq (1 - r) \left( \sum_{n=0}^{N_0} r^n |A_n - A| + \frac{\varepsilon}{2} \sum_{n=N_0+1}^{\infty} r^n \right) \\
\leq (1 - r) \left( \sum_{n=0}^{N_0} r^n |A_n - A| + \frac{\varepsilon}{2} r^{N_0+1} \frac{1}{1 - r} \right) \\
= (1 - r) \sum_{n=0}^{N_0} r^n |A_n - A| + \frac{\varepsilon}{2} r^{N_0+1} \\
\leq (1 - r) \sum_{n=0}^{N_0} r^n |A_n - A| + \frac{\varepsilon}{2}
\]

where the equality \((\star)\) follows from the identity \( \sum_{n=0}^{\infty} r^n = (1 - r)^{-1} \) when \( |r| < 1 \). Now, the expression \( (1 - r) \sum_{n=0}^{N_0} r^n |A_n - A| \) is a polynomial in \( r \) with real coefficients, which takes on the value 0 when \( r = 1 \). Therefore, by continuity of polynomials, there exists a \( \delta > 0 \) such that, if \( 1 - r < \delta \) holds, we have

\[
(1 - r) \sum_{n=0}^{N_0} r^n |A_n - A| \leq \frac{\varepsilon}{2}.
\]

Therefore, taking \( r \) in the range \( 1 - \delta < r < 1 \), we obtain

\[
\left| \sum_{n=1}^{\infty} r^n a_n - A \right| \leq (1 - r) \sum_{n=0}^{N_0} r^n |A_n - A| + \frac{\varepsilon}{2} \leq \varepsilon,
\]

which shows that \( \lim_{r \to 1-} \sum_{n=1}^{\infty} r^n a_n = A \).

**Exercise 16.** (a) When \( n \geq 3 \), we have the inequalities

\[
1 \leq (\log(n))^2 \leq n^2,
\]

which implies

\[
1 = \lim sup_{n \to \infty} 1 \leq \lim sup_{n \to \infty} (\log(n))^{2/n} \leq \lim sup_{n \to \infty} (n^{1/n})^2 = 1,
\]

and therefore the radius of convergence is 1.

(b) We proceed by using the Ratio Test. We have

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} n + 1 = \infty,
\]

and therefore the radius of convergence is 0.

(c) When \( n \geq 1 \), we have

\[
4^n \leq 4^n + 3n \leq 2 \cdot 4^n,
\]
which implies
\[
\frac{n^2}{2 \cdot 4^n} \leq \frac{n^2}{4^n + 3n} \leq \frac{n^2}{4^n}.
\]
This gives
\[
\frac{1}{4} = \limsup_{n \to \infty} \frac{(n^{1/2})^2}{21^n \cdot 4} \leq \limsup_{n \to \infty} \left( \frac{n^2}{4^n + 3n} \right)^{1/n} \leq \limsup_{n \to \infty} \frac{(n^{1/2})^2}{4} = \frac{1}{4},
\]
which implies that the radius of convergence if 4.

(d) A more precise statement of Stirling’s formula says that there exist two constants \(c\) and \(d\) such that \(0 < c < d\) and, for all \(n \geq 1\), we have
\[
(c n^{n+1/2} e^{-n}) \leq n! \leq (d n^{n+1/2} e^{-n}).
\]
Therefore, we have the inequalities
\[
c^3 n^{3n+3/2} e^{-3n} \leq (n!)^3 \leq d^3 n^{3n+3/2} e^{-3n}
\]
and
\[
c^{3n+1/2} n^{3n+1/2} e^{-3n} \leq (3n)! \leq d^{3n+1/2} n^{3n+1/2} e^{-3n},
\]
or, equivalently,
\[
d^{-1} 3^{-3n-1/2} n^{-3n-1/2} e^{3n} \leq \frac{1}{(3n)!} \leq c^{-1} 3^{-3n-1/2} n^{-3n-1/2} e^{3n}.
\]
We deduce
\[
\frac{c^3}{d} \frac{n}{3n \sqrt{3}} \leq \frac{(n!)^3}{(3n)!} \leq \frac{d^3}{c} \frac{n}{3n \sqrt{3}}.
\]
which gives
\[
\frac{1}{27} = \limsup_{n \to \infty} \left( \frac{c^3}{d} \frac{n^{1/n}}{3^n \sqrt{3}^{1/n}} \right)^{1/n} \leq \limsup_{n \to \infty} \left( \frac{(n!)^3}{(3n)!} \right)^{1/n} \leq \limsup_{n \to \infty} \left( \frac{d^3}{c} \frac{n^{1/n}}{3^n \sqrt{3}^{1/n}} \right) = \frac{1}{27}.
\]
Therefore the radius of convergence is 27.

(e) Note first that if \(\alpha\) or \(\beta\) is a negative integer, then the series has only finitely many nonzero terms, and thus converges on all of \(\mathbb{C}\). Assume now that \(\alpha, \beta \notin \{0, -1, -2, \ldots\}\). We again use the ratio test. Let
\[
a_n = \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1) \beta(\beta + 1) \cdots (\beta + n - 1)}{n! \gamma(\gamma + 1) \cdots (\gamma + n - 1)};
\]
we then have
\[
a_{n+1} = \frac{(\alpha + n)(\beta + n)}{(n + 1)(\gamma + n)}.
\]
Now choose \(n \geq |\gamma|\); the triangle equality and reverse triangle inequality give us
\[
\frac{(n-|\alpha|)(n-|\beta|)}{(n+1)(n+|\gamma|)} \leq \left| \frac{(\alpha + n)(\beta + n)}{(n + 1)(\gamma + n)} \right| \leq \frac{(n+|\alpha|)(n+|\beta|)}{(n+1)(n-|\gamma|)}.
\]
Therefore, we get
\[
1 = \limsup_{n \to \infty} \frac{(n-|\alpha|)(n-|\beta|)}{(n+1)(n+|\gamma|)} \leq \limsup_{n \to \infty} \left| \frac{(\alpha + n)(\beta + n)}{(n + 1)(\gamma + n)} \right| \leq \limsup_{n \to \infty} \frac{(n+|\alpha|)(n+|\beta|)}{(n+1)(n-|\gamma|)} = 1,
\]
and the radius of convergence is 1.
(f) If we let \( a_n = \frac{(-1)^n}{n! (n+r)! 4^n} \), we get
\[
\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)(n+r+1)4}.
\]
thus,
\[
\limsup_{n \to \infty} \left| -\frac{1}{(n+1)(n+r+1)4} \right| = 0,
\]
and the radius of convergence is \( \infty \) (that is, the function converges on the entire complex plane).

**Exercise 23.** A function is infinitely differentiable if it has derivatives of all orders. Let \( f(x) \) be defined by
\[
f(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
e^{-1/x^2} & \text{if } x > 0.
\end{cases}
\]
It is clear that if \( x \neq 0 \), then \( f \) has derivatives of all orders at \( x \), and is continuous in a neighborhood of \( x \); it therefore remains to examine the situation at \( x = 0 \). Note first that we may compute the derivatives away from 0 by using the chain rule, and we obtain
\[
f^{(n)}(x) = \begin{cases} 
0 & \text{if } x < 0, \\
e^{-1/x^2} P_n(x) & \text{if } x > 0,
\end{cases}
\]
where \( P_n(x) \) is a polynomial in \( \frac{1}{x} \) (that is, \( P_n(x) = \sum_{n=1}^{N} a_n x^n \)). By making the change of variables \( t = 1/h \), we obtain
\[
\lim_{h \to 0^+} e^{-1/x^2} P_n(x) = \lim_{t \to \infty} \frac{\sum_{n=1}^{N} a_n t^n}{e^{t^2}}.
\]
The expression inside the limit is a linear combination of functions of the form \( \frac{t^k}{e^{t^2}} \); applying L'Hôpital’s rule several times, we obtain
\[
\lim_{t \to \infty} \frac{t^k}{e^{t^2}} = \lim_{t \to \infty} \frac{k!}{e^{t^2} Q_k(t)} = 0,
\]
where \( Q_k(t) \) is some polynomial in \( t \). Therefore, we conclude that
\[
\lim_{h \to 0^+} e^{-1/x^2} P_n(x) = 0.
\]

We must now compute the derivative at 0. We use the definition of derivative from the left and right:
\[
\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{0 - 0}{h} = 0,
\]
\[
\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{e^{-1/h^2}}{h}
= \lim_{t \to \infty} \frac{t}{e^{t^2}}
= \star \lim_{t \to \infty} \frac{1}{2te^{t^2}}
= 0,
\]
where we have used L'Hôpital's rule in (♀). Therefore, \( f'(0) = 0, \) \( f \) is differentiable, and has a continuous first derivative.

Assume now by induction that \( f^{(n-1)}(0) = 0. \) We have an expression for \( f^{(n)} \) away from 0, and at 0 we again use the definitions:

\[
\lim_{h \to 0^-} \frac{f^{(n-1)}(h) - f^{(n-1)}(0)}{h} = \lim_{h \to 0^-} \frac{0 - 0}{h} = 0,
\]

\[
\lim_{h \to 0^+} \frac{f^{(n-1)}(h) - f^{(n-1)}(0)}{h} = \lim_{h \to 0^+} \frac{e^{-1/h^2} P_{n-1}(h)}{h} = \lim_{t \to \infty} \frac{t P_{n-1}(t^{-1})}{e^{t^2}} = 0,
\]

where the last line follows from a computation similar to the one above. Therefore, we conclude that \( f^{(n)}(0) = 0 \) for all \( n \geq 1, \) and therefore \( f \) is infinitely differentiable. Moreover, \( f \) does not have a converging power series expansion at 0, for if it did, we would necessarily have

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0,
\]

which is absurd.

**Exercise 25.** (a) We may evaluate this integral directly. Let \( \gamma \) be parametrized by \( z(t) = Re^{it}, \ t \in [0, 2\pi]. \) If \( n \neq -1, \) we have

\[
\int_{\gamma} z^n dz = \int_0^{2\pi} R^n e^{int} iRe^{it} \ dt = iR^{n+1} \int_0^{2\pi} e^{i(n+1)t} \ dt = iR^{n+1} \left[ \frac{1}{i(n+1)} e^{i(n+1)t} \right]_0^{2\pi} = 0;
\]

if \( n = -1, \) we have

\[
\int_{\gamma} z^{-1} dz = \int_0^{2\pi} R^{-1} e^{-it} iRe^{it} \ dt = i \int_0^{2\pi} dt = 2\pi i.
\]

(If you’re still feeling shaky about these computations, break this up into real and imaginary parts).

(b) We can use Corollary 3.3: if \( \gamma \) does not enclose the origin, then \( f(z) = z^n \) has a primitive defined on the interior of \( \gamma. \) For \( n \neq -1, \) then \( \frac{1}{n+1} z^{n+1} \) is a primitive, defined inside \( \gamma \) (note that this also works for negative exponents). If \( n = -1, \) then \( F(re^{i\theta}) = \log(r) + i\theta \)
is a primitive for \( f(z) = z^{-1} \) (provided we restrict the domain of \( \theta \)). For example, if \( \gamma \) lies entirely in the right half plane, we take \( \theta \in [-\pi, \pi] \). Corollary 3.3 now implies
\[
\int_{\gamma} z^n \, dz = 0.
\]

(c) Using partial fractions, we have
\[
\frac{1}{(z - a)(z - b)} = \frac{1}{a - b} \frac{1}{z - a} + \frac{1}{b - a} \frac{1}{z - b}.
\]
Therefore, if we use part (b) and a slight generalization of part (a), we get
\[
\int_{\gamma} \frac{1}{(z - a)(z - b)} \, dz = \frac{1}{a - b} \int_{\gamma} \frac{1}{z - a} \, dz + \frac{1}{b - a} \int_{\gamma} \frac{1}{z - b} \, dz = \frac{2\pi i}{a - b}.
\]

Chapter 2

Exercise 1. Let \( f(z) = e^{-\gamma z} \), and let \( \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \), where \( \gamma_1 \) is the straight line from 0 to \( R \), \( \gamma_2 \) is the arc \( z(t) = Re^{it}, t \in [0, \frac{\pi}{4}] \), and \( \gamma_3 \) is the straight line from \( R \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \) to 0.

Note first that, since \( f(z) \) is holomorphic, Cauchy’s theorem gives
\[
\int_{\gamma_1} e^{-\gamma z} \, dz + \int_{\gamma_2} e^{-\gamma z} \, dz + \int_{\gamma_3} e^{-\gamma z} \, dz = \int_{\gamma} e^{-\gamma z} \, dz = 0.
\]

Consider first the integral over \( \gamma_2 \). We must show rigorously that this integral goes to 0 as \( R \) goes to \( \infty \). We split the path into an upper and lower part: fix a positive real number \( h \), and let \( \gamma_L \) denote the lower part of \( \gamma_2 \) with \( y \) coordinate less than \( \frac{R}{\sqrt{2}} - h \), and let \( \gamma_U \) denote the upper part of \( \gamma_2 \) with \( y \) coordinate greater than \( \frac{R}{\sqrt{2}} - h \) (draw a picture to see what this looks like). We will make use of the formula of Proposition 3.1(iii) which states
\[
\left| \int_{\gamma} f(z) \, dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma).
\]

We have \( |f(z)| = |e^{-\gamma z}| = e^{-R \Re(\gamma z)} = e^{-z^2 + y^2} \), and on \( \gamma_2 \), we have \( x \geq \frac{R}{\sqrt{2}} \), and therefore \( |e^{-z^2}| \leq e^{-R^2 / 2} e^{y^2} \) on \( \gamma \). Now, on \( \gamma_L \), the function \( |e^{-z^2}| \) is bounded by \( e^{-R^2 / 2} e^{(R / \sqrt{2} - h)^2} = e^{-Rh \sqrt{2} + h^2} \), and the length is bounded by \( R \frac{\pi}{4} \) (the full length of \( \gamma_2 \)). On \( \gamma_U \), \( |e^{-z^2}| \) is bounded by \( e^{-R^2 / 2} e^{R^2 / 2} = 1 \), and the length is bounded by \( \frac{\pi}{2} h \) (notice that the length of \( \gamma_U \) is bounded by the arc length of the quarter circle of radius \( h \)).

Combining everything, we obtain
\[
\left| \int_{\gamma_2} e^{-z^2} \, dz \right| \leq \left| \int_{\gamma_L} e^{-z^2} \, dz \right| + \left| \int_{\gamma_U} e^{-z^2} \, dz \right| \leq \frac{\pi}{4} Re^{-Rh \sqrt{2} + h^2} + \frac{\pi}{2} h.
\]

If we let \( h = \frac{1}{\sqrt{R}} \), this gives
\[
\left| \int_{\gamma_2} e^{-z^2} \, dz \right| \leq \frac{\pi}{4} Re^{-\sqrt{2}R + 1/R} + \frac{\pi}{2 \sqrt{R}},
\]
which tends to 0 as \( R \) goes to infinity.
We now examine the integral over $\gamma_3$. The inverse path $-\gamma_3$ can be parametrized by $z(t) = t \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right), t \in [0, R]$, and therefore

$$
- \int_{\gamma_3} e^{-z^2} \, dz = \int_{\gamma_3} e^{-z^2} \, dz
= \int_0^R e^{-t^2} \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \, dt
= \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \left( \int_0^R \cos(t^2) - i \sin(t^2) \, dt \right).
$$

To complete the computation, we let $R$ tend to infinity, which shows

$$
\int_0^\infty e^{-x^2} \, dx = - \int_{\gamma_3} e^{-z^2} \, dz = \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \left( \int_0^\infty \cos(t^2) - i \sin(t^2) \, dt \right).
$$

This implies

$$
\int_0^\infty \cos(t^2) - i \sin(t^2) \, dt = \left( \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \left( \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right);
$$

comparing real and imaginary parts shows

$$
\int_0^\infty \cos(t^2) \, dt = \int_0^\infty \sin(t^2) \, dt = \frac{\sqrt{2\pi}}{4}.
$$

**Exercise 2.** Consider the function $f(z) = \frac{1}{2i} \frac{e^{iz} - 1}{z}$, and note that, since $f(z) = \frac{1}{2} \sum_{n=1}^\infty \frac{(iz)^n}{n!}$, this function is holomorphic on all of $\mathbb{C}$. We let $\gamma_\varepsilon$ denote the contour for which $-\gamma_\varepsilon$ is parametrized by $z(t) = \varepsilon e^{it}, t \in [0, \pi]$, and let $\gamma_R$ denote the contour $z(t) = R \varepsilon e^{it}, t \in [0, \pi]$. Then Cauchy’s theorem gives

$$
\int_{[-R,-\varepsilon]} \frac{1}{2i} \frac{e^{iz} - 1}{z} \, dz + \int_{\gamma_\varepsilon} \frac{1}{2i} \frac{e^{iz} - 1}{z} \, dz + \int_{[\varepsilon,R]} \frac{1}{2i} \frac{e^{iz} - 1}{z} \, dz + \int_{\gamma_R} \frac{1}{2i} \frac{e^{iz} - 1}{z} \, dz = 0.
$$

We evaluate each of these integrals separately. Consider first the integral over $\gamma_\varepsilon$. Since $f(z)$ is holomorphic, the quantity $|f(z)|$ is bounded by some $M$ in some small disc of radius $\varepsilon_0$. Taking $\varepsilon < \varepsilon_0$, we obtain

$$
\left| \int_{\gamma_\varepsilon} \frac{1}{2i} \frac{e^{iz} - 1}{z} \, dz \right| \leq M \cdot \text{length}(\gamma_\varepsilon) = M \varepsilon \pi.
$$

Note that as long as $\varepsilon < \varepsilon_0$, the bound on $\sup_{z \in \gamma_\varepsilon} |f(z)|$ is independent of $\varepsilon$. Therefore, letting $\varepsilon$ tend to 0, we obtain

$$
\int_{\gamma_\varepsilon} \frac{1}{2i} \frac{e^{iz} - 1}{z} \, dz \to 0.
$$

We now consider the integral over $\gamma_R$, again by splitting $\gamma_R$ into a lower and upper part. We again fix a positive real number $h$, and let $\gamma_L$ denote the part of $\gamma_R$ with $y$ coordinate less than $h$, and let $\gamma_U$ denote the part of $\gamma_R$ with $y$ coordinate greater than $h$. We have $|e^{iz}| = e^{R|iz|} = e^{-y}$, and therefore on $\gamma_L$ the maximum of this function is 1, and on $\gamma_U$ the maximum is $e^{-h}$. The length of $\gamma_L$ is bounded by $\pi h$ (since the length of the two small arc
is certainly bounded above by the length of half-circle of radius \( h \), and the length of \( \gamma_U \) is bounded by \( R\pi \). Therefore, we obtain
\[
\left| \int_{\gamma_R} \frac{1}{2i} \frac{e^{iz}}{z} \, dz \right| = \left| \int_{\gamma_L} \frac{1}{2i} \frac{e^{iz}}{z} \, dz + \int_{\gamma_U} \frac{1}{2i} \frac{e^{iz}}{z} \, dz \right|
\leq \left| \int_{\gamma_L} \frac{1}{2i} \frac{e^{iz}}{z} \, dz \right| + \left| \int_{\gamma_U} \frac{1}{2i} \frac{e^{iz}}{z} \, dz \right|
\leq \frac{1}{2R} \pi h + \frac{1}{2} e^{-h} R \pi
= \frac{\pi h}{2R} + \frac{\pi e^{-h}}{2}.
\]
Therefore, taking \( h = \sqrt{R} \), we get
\[
\left| \int_{\gamma_R} \frac{1}{2i} \frac{e^{iz}}{z} \, dz \right| \leq \frac{\pi}{2\sqrt{R}} + \frac{\pi e^{-\sqrt{R}}}{2},
\]
which tends to 0 as \( R \) goes to infinity. On the other hand, using the parametrization \( z(t) = Re^{it} \), we get
\[
\int_{\gamma_R} \frac{1}{2iz} \, dz = \frac{1}{2} \int_0^\pi dt = \frac{\pi}{2}.
\]
Therefore, as \( R \) goes to infinity, we obtain
\[
\int_{\gamma_R} \frac{1}{2i} \frac{e^{iz} - 1}{z} \, dz \longrightarrow -\frac{\pi}{2}.
\]
Upon combining everything and letting \( R \) tend to infinity and \( \varepsilon \) tend to 0, we get
\[
\int_{-\infty}^{\infty} \frac{1}{2i} \frac{e^{ix} - 1}{x} \, dx = \frac{\pi}{2};
\]
taking the real part of both sides shows
\[
\int_0^\infty \frac{\sin(x)}{x} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \, dx = \frac{\pi}{2}.
\]
Remark. Let \( P(z) \) and \( Q(z) \) be two polynomials such that \( \deg(P(z)) + 1 \leq \deg(Q(z)) \). Then the above proof actually shows that we have
\[
\int_{\gamma_R} \frac{P(z)}{Q(z)} e^{iz} \, dz \longrightarrow 0
\]
as \( R \) tends to infinity.

**Exercise 3.** Let \( a, b \) be two positive real numbers (we can always assume \( b \) is positive), let \( A = \sqrt{a^2 + b^2} \), and let \( 0 \leq \omega \leq \frac{\pi}{2} \) be the unique angle for which \( \cos(\omega) = \frac{a}{A} \). Note that this implies \( \sin(\omega) = \frac{b}{A} \). Let \( \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \), where \( \gamma_1 \) is the straight line from 0 to \( R \), \( \gamma_2 \) is the
curve parametrized by $z(t) = Re^{it}$, $t \in [0, \omega]$, and $\gamma_3$ is the straight line from $Re^{i\omega}$ to 0. By Cauchy’s theorem, we have

$$\int_{\gamma_1} e^{-Az} \, dz + \int_{\gamma_2} e^{-Az} \, dz + \int_{\gamma_3} e^{-Az} \, dz = \int_{\gamma} e^{-Az} \, dz = 0. $$

We first evaluate the integral along $\gamma_2$. We have

$$\int_{\gamma_2} e^{-Az} \, dz = \left[ -\frac{e^{-Az}}{A} \right]_{z=Re^{i\omega}}^{z=R} = e^{-aR} - e^{-ARRe^{i\omega}} = e^{-AR} - e^{-R(a+ib)} = \frac{e^{-aR} + e^{-Ra}}{A};$$

therefore

$$\left| \int_{\gamma_2} e^{-Az} \, dz \right| \leq \frac{e^{-aR} + e^{-Ra}}{A},$$

which tends to 0 as $R$ tends to infinity.

Thus, letting $R$ tend to infinity, we obtain

$$\int_{\gamma_3} e^{-Az} \, dz = \int_{0}^{\infty} (\cos(\omega) + i \sin(\omega))e^{-A(\cos(\omega) + i \sin(\omega))t} \, dt$$

$$= \int_{0}^{\infty} \left( \frac{a}{A} + i \frac{b}{A} \right) e^{-at} \, dt$$

$$= \int_{0}^{\infty} e^{-at} \, dt$$

$$= \left[ \frac{e^{-at}}{A} \right]_{t=0}^{\infty}$$

$$= \frac{1}{A}.$$ 

Equating real and imaginary parts (and using the relations $\cos(\omega) = \frac{a}{A}$ and $\sin(\omega) = \frac{b}{A}$), we get the pair of equations

$$\frac{a}{A} \int_{0}^{\infty} e^{-at} \cos(bt) \, dt + \frac{b}{A} \int_{0}^{\infty} e^{-at} \sin(bt) \, dt = \frac{1}{A}$$

$$\frac{b}{A} \int_{0}^{\infty} e^{-at} \cos(bt) \, dt - \frac{a}{A} \int_{0}^{\infty} e^{-at} \sin(bt) \, dt = 0.$$ 

Writing this in matrix form, we get

$$\begin{pmatrix} \frac{a}{A} & \frac{b}{A} \\ \frac{b}{A} & -\frac{a}{A} \end{pmatrix} \begin{pmatrix} \int_{0}^{\infty} e^{-at} \cos(bt) \, dt \\ \int_{0}^{\infty} e^{-at} \sin(bt) \, dt \end{pmatrix} = \begin{pmatrix} \frac{1}{A} \\ 0 \end{pmatrix}.$$
Inverting this, we get

\[
\int_0^\infty e^{-at} \cos(bt) \, dt = \frac{a}{A^2} = \frac{a}{a^2 + b^2},
\]

\[
\int_0^\infty e^{-at} \sin(bt) \, dt = \frac{b}{A^2} = \frac{b}{a^2 + b^2}.
\]