Chapter 15 - Integration

1. Compute the integral 

\[ \iint_D xy \, dxdy, \]

where \( D \) is described by \( 1 \leq x^2 + y^2 \leq 4 \) and \( 0 \leq y \leq x \).

**Solution.** The region \( D \) in polar coordinates is given by 

\[ 1 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{4}, \]

so the integral becomes

\[
\int_0^{\pi/4} \int_1^2 r^2 \cos(\theta) \sin(\theta) \, r \, dr \, d\theta = \frac{1}{4} \int_0^{\pi/4} \left[ r^4 \cos(\theta) \sin(\theta) \right]_{r=1}^{r=2} \, d\theta = \frac{15}{4} \int_0^{\pi/4} \cos(\theta) \sin(\theta) \, d\theta \\
= \frac{15}{4} \left[ -\frac{\cos(2\theta)}{4} \right]_{\theta=0}^{\theta=\pi/4} = \frac{15}{16}.
\]

2. Evaluate the following double integral

\[ \int_0^8 \int_{3\sqrt{y}}^{2} y^2 e^{x^2} \, dx \, dy \]

by switching the order of integration.

**Solution.** The given boundaries of integration are

\[ 0 \leq y \leq 8, \quad 3\sqrt{y} \leq x \leq 2, \]

which is a region of type II in the \( xy \)-plane. Expressing it as a type I region, we get

\[ 0 \leq x \leq 2, \quad 0 \leq y \leq x^3. \]

The integral then becomes

\[
\int_0^2 \int_0^{x^3} \frac{y^2 e^{x^2}}{x^8} \, dy \, dx = \frac{1}{3} \int_0^{x^3} xe^{x^2} \, dx = \frac{1}{3} \left[ \frac{e^{x^2}}{2} \right]_{x=0}^{x=2} = \frac{e^4}{6} - \frac{1}{6}.
\]

3. Find the total mass of the solid \( E \) which lies above the cone \( z^2 = 4x^2 + 4y^2 \) with \( z \geq 0 \) and below the plane \( z = 4 \), where the density is given by \( \rho(x, y, z) = k \), with \( k \) constant.
Recall that the total mass is
\[ M = \iiint_E \rho(x, y, z) \, dV. \]

**Solution.** We use cylindrical coordinates. The projection of the solid \( E \) onto the \( xy \)-plane is the disc with center the origin and radius 2 (notice that intersecting the plane with the cone, we get \( 16 = 4x^2 + 4y^2 \)). In polar coordinates this region is described by
\[ 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi. \]

Above this projection, the \( z \) component goes from the surface of the cone \( z = \sqrt{4x^2 + 4y^2} = 2r \) up to the plane \( z = 4 \). The total mass of \( E \) is therefore given by
\[ k \int_0^{2\pi} \int_0^2 \int_0^4 r \, dz \, dr \, d\theta = k \int_0^{2\pi} \int_0^2 (4r - 2r^2) \, dr \, d\theta = 2\pi k \left[ 2r^2 - \frac{2r^3}{3} \right]_{r=0}^{r=2} = \frac{64\pi k}{12} = \frac{16\pi k}{3}. \]

4. Compute the triple integral
\[ \iiint_E x^2 \, dx \, dy \, dz, \]
where \( E \) is the tetrahedron with vertices \((0,0,0),(1,0,0),(0,2,0),(0,0,3)\).

**Solution.** We write the tetrahedron \( E \) as a region of type 1 in \( \mathbb{R}^3 \). Its projection \( D \) is the triangle in the \( xy \)-plane described by
\[ 0 \leq x \leq 1, \quad 0 \leq y \leq 2 - 2x. \]

The tetrahedron \( E \) lies above \( D \) between the graph of the function \( u_1(x,y) = 0 \) and the plane passing through the points \((1,0,0),(0,2,0),(0,0,3)\). To write down the equation of this plane, we choose two vectors on this plane \((-1,0,3),(-1,2,0)\) and compute their cross product, which is \((-6,-3,-2)\). It follows that the equation of the plane is
\[ 6x + 3y + 2z = 6. \]

Solving for \( z \), we see that the plane is the graph of the function
\[ u_2(x,y) = 3 - \frac{3y}{2} - 3x. \]

The triple integral then becomes
\[ \int_0^1 \int_0^{2-2x} \int_0^{3-3y/2-3x} x^2 \, dz \, dy \, dx = \int_0^1 \int_0^{2-2x} \left( 3x^2 - \frac{3x^2y}{2} - 3x^3 \right) \, dy \, dx = \int_0^1 \left( 3x^2 - 6x^3 + 3x^4 \right) \, dx \\
= \left[ x^3 - \frac{3x^4}{2} + \frac{3x^5}{5} \right]_{x=0}^{x=1} = \frac{1}{10}. \]

5. Calculate
\[ \int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin(x) \, dz \, dy \, dx. \]
Solution. The bounds are given to us; we simply compute. We have
\[
\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin(x) \, dz \, dy \, dx = \int_0^\pi \int_0^1 y \sqrt{1-y^2} \sin(x) \, dy \, dx
\]
\[
= \left( \int_0^\pi \sin(x) \, dx \right) \left( \int_0^1 y \sqrt{1-y^2} \, dy \right)
\]
\[
= \left. \left[ -\cos(x) \right]_{x=0}^{x=\pi} \cdot \left[ -\frac{1}{3} (1-y^2)^{3/2} \right]_{y=1}^{y=0}
\]
\[
= \frac{2}{3}.
\]

6. Compute
\[
\iiint_E x^2 y \, dx \, dy \, dz,
\]
where \( E \) is the region in \( xyz \)-space bounded by the paraboloid \( z = 1 - x^2 - y^2 \) and the plane \( z = 0 \).

Solution. The projection of \( D \) onto the \( xy \)-plane is a circle of radius 1, centered at the origin. Therefore, we may express our region \( E \) in cylindrical coordinates by
\[
0 \leq r \leq 1
\]
\[
0 \leq \theta \leq 2\pi
\]
\[
0 \leq z \leq 1 - r^2
\]
In cylindrical coordinates, the integral becomes
\[
\iiint_E x^2 y \, dx \, dy \, dz = \int_0^1 \int_0^{2\pi} \int_0^{1-r^2} r^4 \cos(\theta)^2 \sin(\theta)^2 r \, dz \, d\theta \, dr
\]
\[
= \int_0^1 \int_0^{2\pi} (r^5 - r^7) \cos(\theta)^2 \sin(\theta)^2 \, d\theta \, dr
\]
\[
= \left( \int_0^1 r^5 - r^7 \, dr \right) \left( \int_0^{2\pi} \frac{1+\cos(2\theta)}{2} \frac{1-\cos(2\theta)}{2} \, d\theta \right)
\]
\[
= \left[ \frac{1}{6} r^6 - \frac{1}{8} r^8 \right]_{r=0}^{r=1} \left( \int_0^{2\pi} \frac{1-\cos(2\theta)^2}{4} \, d\theta \right)
\]
\[
= \frac{1}{24} \left( \int_0^{2\pi} \frac{1}{4} - \frac{1}{8} \cos(4\theta) \, d\theta \right)
\]
\[
= \frac{1}{24} \left[ \frac{\theta}{8} - \frac{\sin(4\theta)}{32} \right]_{\theta=0}^{\theta=2\pi}
\]
\[
= \frac{\pi}{96}.
\]

Chapter 16 - Fundamental Theorem of Line Integrals, Green’s, Stokes’, Divergence

1. Compute the line integral of the vector field
\[
\vec{F}(x, y, z) = \langle \cos(x), 2 + \cos(y), e^z \rangle,
\]
along the spiral curve $C$ parametrized by $\vec{r}(t) = \langle t, \cos(t), \sin(t) \rangle$ with $0 \leq t \leq 3\pi$. (Hint: is $\vec{F}$ conservative?)

**Solution.** It is straightforward to check that $\nabla \times \vec{F} = \vec{0}$, and since $\vec{F}$ is defined everywhere, it is conservative $\vec{F} = \nabla f$. We can easily compute a potential function $f$ (for example by partial integration, or in this case just guessing)

$$f(x, y, z) = \sin(x) + 2y + \sin(y) + e^z.$$

By the Fundamental Theorem for Line Integrals

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(3\pi)) - f(\vec{r}(0)) = f(3\pi, -1, 0) - f(0, 1, 0) = -2 + \sin(-1) + 2 - \sin(1) - 1 = -4 - 2 \sin(1).$$

2. Compute the surface area of the surface $S$ parametrized by

$$\vec{r}(u, v) = \left\langle 3u + 2v, 4u + v, \frac{2}{7} v^{7/2} \right\rangle,$$

with $0 \leq u \leq 1$ and $u^{1/4} \leq v \leq 1$.

**Solution.** We compute

$$\vec{r}_u = \langle 3, 4, 0 \rangle,$$
$$\vec{r}_v = \langle 2, 1, v^{5/2} \rangle,$$
$$\vec{r}_u \times \vec{r}_v = \langle 4v^{5/2}, -3v^{5/2}, -5 \rangle,$$
$$||\vec{r}_u \times \vec{r}_v|| = \sqrt{25v^5 + 25} = 5\sqrt{v^5 + 1}.$$

The surface area is then given by

$$5 \int_0^1 \int_{u^{1/4}}^1 \sqrt{v^5 + 1} \, dv \, du,$$

which is impossible to compute, so we have to switch the order of integration. The given boundaries of integration are

$$0 \leq u \leq 1,$$
$$u^{1/4} \leq v \leq 1,$$

which is a region of type I in the $uv$-plane. Expressing it as type II we get

$$0 \leq v \leq 1,$$
$$0 \leq u \leq v^4.$$

The integral then becomes

$$5 \int_0^1 \int_0^{v^4} \sqrt{v^5 + 1} \, du \, dv = 5 \int_0^1 v^4 \sqrt{v^5 + 1} \, dv = 5 \left[ \frac{2(v^5 + 1)^{3/2}}{5 \cdot 3} \right]_{v=0}^{v=1} = \frac{4\sqrt{2}}{3} - \frac{2}{3}.$$
3. Find the flux of the vector field

\[ \vec{F}(x, y, z) = (5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z)) \]

through the surface \( S \) of the sphere with center the origin, radius 3 and outward orientation. (Hint: Divergence Theorem).

**Solution.** The divergence of \( \vec{F} \) is

\[ \nabla \cdot \vec{F} = 15x^2 + 15y^2 + 15z^2, \]

and so by the Divergence Theorem, the flux equals the triple integral

\[ \iiint_E 15(x^2 + y^2 + z^2) \, dxdydz, \]

where \( E \) is the ball with center the origin and radius 3. We use spherical coordinates and get

\[ 15 \int_0^\pi \int_0^{2\pi} \int_0^3 \rho^2 \cdot \rho^2 \sin(\phi) \, d\rho d\theta d\phi = 3 \cdot 3^5 \cdot 2\pi \cdot 2 = 2916\pi. \]

4. Use Stokes' Theorem to find the flux

\[ \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}, \]

where

\[ \vec{F}(x, y, z) = (xz^3 e^{y^2}, 2xyze^{x^2+z}, x + z^3 e^{x^2}, ye^{x^2+z} + ze^x), \]

and the surface \( S \) is the upper hemisphere \( x^2 + y^2 + z^2 = 4 \) with \( z \geq 0 \) and upward orientation (notice that the surface \( S \) is not closed).

**Solution.** We use Stokes' Theorem to convert the flux integral to

\[ \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}, \]

where \( C \) is the circle \( x^2 + y^2 = 4 \) in the \( xy \)-plane \( z = 0 \), with the counterclockwise orientation. We can parametrize \( C \) by

\[ \vec{r}(t) = (2\cos(t), 2\sin(t), 0), \]

with \( 0 \leq t \leq 2\pi \), so the line integral equals

\[ \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt. \]

We compute

\[ \vec{F}(\vec{r}(t)) = (0, 2\cos(t), 2\sin(t)e^{4\cos(t)^2}), \]

\[ \vec{r}'(t) = (-2\sin(t), 2\cos(t), 0), \]

\[ \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 4\cos(t)^2. \]
The original flux integral is then equal to
\[ 4 \int_0^{2\pi} \cos(t)^2 \, dt = 4\pi. \]

5. Find the flux
\[ \iint_S \vec{F} \cdot d\vec{S}, \]
where
\[ \vec{F}(x, y, z) = \langle xz, x, y \rangle \]
and \( S \) is the hemisphere \( x^2 + y^2 + z^2 = 1, y \geq 0 \), with the normal vector pointing outwards.

**Solution.** We may compute this directly. We parametrize the surface \( S \) by
\[ \vec{r}(\theta, \phi) = \langle \cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi) \rangle, \]
where we take \( 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq \pi \) in order to get the right half of the sphere. We’ve computed in class that the normal vector pointing outwards is given by
\[ \vec{r}_\phi \times \vec{r}_\theta = \sin(\phi)^2 \cos(\theta) \hat{i} + \sin(\phi)^2 \sin(\theta) \hat{j} + \sin(\phi) \cos(\phi) \hat{k}. \]
We have
\[ \vec{F}(\vec{r}(\theta, \phi)) = \sin(\phi) \cos(\phi) \cos(\theta) \hat{i} + \sin(\phi) \cos(\theta) \hat{j} + \sin(\phi) \sin(\theta) \hat{k} \]
\[ \vec{F}(\vec{r}(\theta, \phi)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) = \sin(\phi)^3 \cos(\phi) \cos(\theta)^2 + \sin(\phi)^3 \sin(\theta) \cos(\theta) + \sin(\phi)^2 \cos(\phi) \sin(\theta) \]
Thus, the flux is
\[ \iint_S \vec{F} \cdot d\vec{S} = \int_0^\pi \int_0^\pi \vec{F}(\vec{r}(\theta, \phi)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) \, d\theta d\phi \]
\[ = \int_0^\pi \int_0^\pi \sin(\phi)^3 \cos(\phi) \cos(\theta)^2 \, d\theta d\phi + \int_0^\pi \int_0^\pi \sin(\phi)^3 \sin(\theta) \cos(\theta) \, d\theta d\phi \]
\[ + \int_0^\pi \int_0^\pi \sin(\phi)^2 \cos(\phi) \sin(\theta) \, d\theta d\phi \]
\[ = \left( \int_0^\pi \sin(\phi)^3 \cos(\phi) d\phi \right) \left( \int_0^\pi \cos(\theta)^2 d\theta \right) + \left( \int_0^\pi \sin(\phi)^3 d\theta \right) \left( \int_0^\pi \sin(\theta) \cos(\theta) d\theta \right) \]
\[ + \left( \int_0^\pi \sin(\phi)^2 d\phi \right) \left( \int_0^\pi \sin(\theta) d\theta \right) \]
\[ = 0. \]

6. Find the work done by the vector field
\[ F(x, y) = \langle e^{x-1}, xy \rangle \]
on a particle moving along the path \( \vec{r}(t) = \langle t^2, \ t^3 \rangle, \ 0 \leq t \leq 1. \)

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Solution. We evaluate directly. We have
\[
\vec{r}'(t) = \langle 2t, 3t^2 \rangle
\]
\[
\vec{F}(\vec{r}(t)) = \langle e^{t^2-1}, t^5 \rangle
\]
\[
\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 2te^{t^2-1} + 3t^7
\]

Hence
\[
\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]
\[
= \int_0^1 2te^{t^2-1} + 3t^7 \, dt
\]
\[
= \left[ e^{t^2-1} + \frac{3}{8}t^8 \right]_0^1
\]
\[
= \frac{11}{8} - e^{-1}.
\]

7. Evaluate the line integral
\[
\oint_C y^2 \, dx + x^2 \, dy,
\]
where \(C\) is the closed curve which is the boundary of the triangle with vertices \((0,0), (1,1)\) and \((1,0)\), with the counterclockwise orientation.

Solution. We could evaluate this directly, but it’s easier to use Green’s Theorem. This says that
\[
\oint_C P(x,y) \, dx + Q(x,y) \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy,
\]
where \(P(x,y) = y^2\), \(Q(x,y) = x^2\) and \(D\) is the inside of the triangle. Clearly
\[
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2y.
\]

We can describe the triangle \(D\) by
\[
0 \leq x \leq 1, \quad 0 \leq y \leq x,
\]
and so the line integral is equal to
\[
\int_0^1 \int_0^x (2x - 2y) \, dy \, dx = \int_0^1 \left( - \frac{y^2}{2} + xy \right)_{y=0}^{y=x} \, dx = \int_0^1 x^2 \, dx = \frac{1}{3}.
\]

8. Recall the following consequence of Green’s Theorem (that we saw in class). If a simple closed curve \(C\) in the \(xy\)-plane encloses a region \(D\), with positive orientation, then the area of \(D\) is equal to the line integral
\[
\oint_C x \, dy.
\]
(Remind yourself, using Green's Theorem, why this gives the area.) Use this to compute the area inside the Lemniscate of Gerono (see Wikipedia), which has equation \( x^4 = x^2 - y^2 \) and is parametrized by
\[
\vec{r}(t) = (\sin(t),\ \sin(t) \cos(t)),
\]
with \( 0 \leq t \leq \pi \).

Solution. We need to compute the line integral \(- \int_C x\,dy\) (the minus sign is because the given parametrization gives the negative orientation). We compute
\[
\vec{r}'(t) = (\cos(t),\ \cos(t)^2 - \sin(t)^2)
\]
So \( x(t) = \sin(t),\ y(t) = \sin(t) \cos(t),\ x'(t) = \cos(t),\ \text{and}\ y'(t) = \cos(t)^2 - \sin(t)^2 \). The area equals
\[
- \int_0^\pi x(t)y'(t)\,dt = - \int_0^\pi \cos(t)^2 \sin(t) - \sin(t)^3\,dt
\]
\[
= - \int_0^\pi \cos(t)^2 \sin(t) - \sin(t)(1 - \cos(t)^2)\,dt
\]
\[
= - \int_0^\pi 2 \cos(t)^2 \sin(t) - \sin(t)\,dt
\]
\[
= - \left[ \frac{2}{3} \cos(t)^3 + \cos(t) \right]_0^\pi
\]
\[
= \frac{2}{3}.
\]

9. Evaluate
\[
\oint_C y^4\,dx + 2xy^3\,dy,
\]
where \( C \) is the ellipse \( x^2 + 2y^2 = 2 \).

Solution We have \( P(x,y) = y^4 \) and \( Q(x,y) = 2xy^3 \). Both of these functions are defined on the entire \( xy\)-plane, so we may apply Green's Theorem. We have
\[
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y^3 - 4y^3 = -2y^3.
\]
Therefore, we have
\[
\oint_C y^4\,dx + 2xy^3\,dy = \iint_D -2y^3\,dA,
\]

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where $D$ is the region inside the ellipse $x^2 + 2y^2 = 2$. Notice that this region is symmetric about the $x$-axis, and that the function $-2y^3$ is odd with respect to $y$. Therefore, by symmetry, we conclude that

$$\int \int_D -2y^3 \, dA = 0.$$ 

**Complex Analysis**

1. Evaluate in both polar and cartesian coordinates:

   (a) $(-3 + 3i)^{11}$  
   (b) $(-1 + \sqrt{3}i)^6$

**Solution.** We write the base in polar form and use de Moivre’s Theorem.

(a) $(-3 + 3i)^{11} = [3\sqrt{2} \left( \cos \left( \frac{3\pi}{4} \right) + i \sin \left( \frac{3\pi}{4} \right) \right)]^{11} = 3^{11/2} \left( \cos \left( \frac{33\pi}{4} \right) + i \sin \left( \frac{33\pi}{4} \right) \right)$

$$= 3^{11/2} \left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = 3^{11/2} (1 + i).$$

(b) $(-1 + \sqrt{3}i)^6 = \left[ 2 \left( \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \right) \right]^6 = 2^6 \left( \cos \left( \frac{12\pi}{3} \right) + i \sin \left( \frac{12\pi}{3} \right) \right) = 64.$

2. Is the complex function $f(z) = e^z$ holomorphic or not?

**Solution.** We have $z = x - iy$, so $f(z) = e^z = e^x e^{-iy} = e^x \cos(y) - ie^x \sin(y)$. We compute the partial derivatives of the real and imaginary parts:

$$\frac{\partial u}{\partial x} = e^x \cos(y) \quad \frac{\partial v}{\partial y} = -e^x \cos(y)$$

$$\frac{\partial u}{\partial y} = -e^x \sin(y) \quad \frac{\partial v}{\partial x} = e^x \sin(y)$$

Since the Cauchy-Riemann equations aren’t satisfied, $f(z)$ cannot be holomorphic.

3. At which points $z$ is the complex function $f(z) = \frac{1}{z^2 + 1}$ holomorphic? Compute its complex derivative $f'(z)$ at those points.

**Solution.** We know that the function $z^2 + 1$ is holomorphic (it is the sum of $z^2$ and 1, both of which are holomorphic). The quotient

$$\frac{1}{z^2 + 1}$$

is holomorphic wherever the denominator does not vanish. The denominator vanishes precisely when

$$z^2 + 1 = 0,$$
which means $z = i$ or $z = -i$. At these two points the function $f(z)$ is not defined. Everywhere else, it is defined and it is holomorphic. Its complex derivative is computed with the standard quotient rule

$$f'(z) = -(z^2 + 1)^{-2} \cdot 2z = -\frac{2z}{(z^2 + 1)^2}.$$  

4. Let $C$ be the straight segment joining $2i$ to $1 + i$. Compute

$$\int_C z^2 dz.$$  

Solution. We could do this directly, but we can also proceed as follows. The function $z^2$ has antiderivative $F(z) = z^3/3$, so the line integral equals

$$F(1 + i) - F(2i) = \frac{(1 + i)^3}{3} - \frac{(2i)^3}{3} = -\frac{2}{3} + \frac{10}{3}i.$$  

5. Let $C$ be the curve given by

$$z(t) = t \sin(t)e^{1-t-t^2} + \ln(t(2\pi - t) + 1)i, \quad 0 \leq t \leq 2\pi.$$  

Evaluate

$$\int_C \cos(e^z) \, dz.$$  

Solution. The curve $C$ is closed (note that $z(0) = 0 = z(2\pi)$), and the function $\cos(e^z)$ is analytic everywhere (but note that we can’t write down an antiderivative!). Therefore, by Cauchy’s Theorem,

$$\int_C \cos(e^z) \, dz = 0.$$  

6. Let $C$ be the unit circle in the plane centered at the origin and oriented counterclockwise. Evaluate each of the following integrals, giving a concise explanation if you use any Theorem.

(a) $\int_C e^{z^4} \, dz$  
(b) $\int_C \frac{z + i}{z - \frac{i}{2}} \, dz$  
(c) $\int_C \frac{1}{z} \, dz$  
(d) $\int_C \frac{1}{z + 2} \, dz$  
(e) $\int_C (3z - 1)^{-5} \, dz$  

Solution. Call $D$ the unit disk enclosed by $C$.

(a) The function $e^{z^4}$ is holomorphic on $D$, so by Cauchy’s Theorem

$$\int_C e^{z^4} \, dz = 0.$$  

(b) The point $z_0 = i/2$ is inside $D$, so by Cauchy’s Integral Formula

$$\int_C \frac{z + i}{z - \frac{i}{2}} \, dz = 2\pi i \left( i + \frac{i}{2} \right) = -3\pi.$$  

(c) The point $z_0 = 0$ is inside $D$, so by Cauchy’s Integral Formula (with $f(z) = 1$)

$$\int_C \frac{1}{z} \, dz = 2\pi i.$$  

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(d) The function $\frac{1}{z+2}$ is holomorphic on the whole of $D$ (since the denominator does not vanish on $D$), therefore by Cauchy’s Theorem

$$\int_C \frac{1}{z+2} \, dz = 0.$$

(e) The function $(3z - 1)^{-5}$ has antiderivative

$$-\frac{1}{12} (3z - 1)^{-4}$$

which is defined on the curve $C$, which is closed, so

$$\int_C (3z - 1)^{-5} \, dz = 0.$$